

Fibre-wise linear Poisson structures related to W^* -algebras

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Abstract

In this paper we investigate fiber-wise linear complex Banach sub-Poisson structures defined canonically by the structure of a W^* -algebra \mathfrak{M} . In particular we show that these structures are arranged in the short exact sequence of complex Banach sub-Poisson \mathcal{VB} -groupoids with the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of \mathfrak{M} as the side groupoid.

Introduction

Poisson geometry is of fundamental importance for description of properties of finite as well as infinite dimensional classical Hamiltonian systems, [8, 9, 12]. Discovery of the symplectic groupoid over a Poisson manifold, [11, 23, 24, 25], which generalize the cotangent groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ over the Lie-Poisson space \mathfrak{g}^* , where \mathfrak{g}^* is the dual of the Lie algebra \mathfrak{g} of a Lie group G , implemented the Lie groupoid theory to Poisson geometry. On the other hand the natural correspondence between fibre-wise Poisson structure and Lie algebroids allows us to consider the Lie algebroid theory as a part of Poisson geometry. From the above and from the fact that Lie algebroids are the infinitesimal version of Lie groupoids, [9, 13], one can incorporate the theory of them to the geometric methods of classical mechanics.

No less crucial then Poisson geometry for description of the classical physical systems is the theory of operator algebras, especially the theory of W^* -algebras, for the description of quantum physical systems. By definition a W^* -algebra (von Neumann algebra) is a C^* -algebra \mathfrak{M} which has a Banach predual space \mathfrak{M}_* , i.e. $\mathfrak{M} = (\mathfrak{M}_*)^*$, see [20] for the details. This property guarantees that the complete lattice $\mathcal{L}(\mathfrak{M})$ of orthogonal projections from \mathfrak{M} has plenty of elements and, thus allows one to interpret the W^* -algebras theory as non-commutative probability theory and leads to the von Neumann theory of quantum measurement, [10]. In consequence, the propositional calculus of quantum mechanics, called quantum logic, is based on the lattice structure of $\mathcal{L}(\mathfrak{M})$. The quantum observables are defined as σ -homomorphisms from the lattice $\mathcal{B}(\mathbb{R})$ of the Borel sets on the real line into $\mathcal{L}(\mathfrak{M})$, i.e. they are $\mathcal{L}(\mathfrak{M})$ -valued spectral measures. The states of the quantum system which corresponds to \mathfrak{M} are normalized positive elements of \mathfrak{M}_* . The case when the W^* -algebra \mathfrak{M} is the algebra of bounded linear operators $L^\infty(\mathcal{H})$ on the complex separable Hilbert space \mathcal{H} implements a standard model of quantum mechanics.

One of the most intriguing problems of mathematical physics is to describe the passage from the classical Hamiltonian systems to the quantum ones, known as a quantization procedure. Though this question will not be touched upon in the present paper we show, however, that the W^* -algebra structure defines in a natural way a family of complex (real) fibre-wise linear Banach sub-Poisson structures on the complex Banach vector bundles over $\mathcal{L}(\mathfrak{M})$ and over other Banach manifolds related to \mathfrak{M} . As an example of such type of structure one can take the Banach Lie-Poisson structure on \mathfrak{M}_* , which in the case $\mathfrak{M} = L^\infty(\mathcal{H})$ allows to consider Liouville-von Neumann equation as a Hamiltonian equation on $L^1(\mathcal{H}) \cong L^\infty(\mathcal{H})_*$, see [16].

Independently from this physical application the theory of W^* -algebras is one of the crucial topics of contemporary mathematics which interconnects analysis with algebra, [6, 21].

It is rather unexpected that to the category of W^* -algebras corresponds in a functorial way a category of Banach-Lie groupoids and since of that the Banach-Lie algebroids. Namely in [17], the complex Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of \mathfrak{M} was introduced and investigated. The Banach-Lie algebroids corresponding to these groupoids are described in detail in [18]. Here we will continue the investigation of the mentioned structures as well as we will study fibre-wise linear sub-Poisson structures which are related to them in a canonical way. Through out the paper we use the noncommutative coordinates for the description of the structures under investigation. The calculus in these coordinates is based on the combining the algebra structure of \mathfrak{M} with the groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

The structure of this paper is as follows. In Section 1, following [17], we briefly discuss the structure of the complex Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and show that it splits into the transitive Banach-Lie subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$. These subgroupoids are closed-open complex Banach submanifolds of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ isomorphic to the corresponding gauge groupoids, see Proposition 1.2 and Proposition 1.4. Additionally to [17], we prove that $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a Hausdorff topological groupoid with respect to the topology underlying its

complex Banach manifold structure, see Proposition 1.1. In Proposition 1.3 we characterize the set of central projections of \mathfrak{M} in terms of this underlying topology.

The tangent prolongation groupoid $T\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows T\mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is studied in Section 2. The main results of this section are presented in diagrams (2.13), (2.16) and (2.19).

The Banach-Lie algebroid $\mathcal{AG}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and the Atiyah sequence (3.8) of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ are described in Section 3, see Proposition 3.1 and Proposition 3.2. In particular we present the explicit "coordinate" formula for the algebroid Lie bracket and the algebroid anchor map, see Proposition 3.3 and Proposition 3.4.

Generalizing the results of [14] to Banach sub Poisson case in the last two sections of this paper we investigate the fibre-wise linear Poisson structures related to a W^* -algebra. So, in Section 4 we show that the short exact sequence (4.2) predual to the Atiyah sequence (3.8) is a short exact sequence of the fibre-wise linear complex Banach sub Poisson manifolds. The description of their structure, including the structure of foliation on the symplectic leaves, is presented in Proposition 4.2, Theorem 4.4 and Theorem 4.5. The exact sequence (5.21) of the complex Banach sub Poisson \mathcal{VB} -groupoids with $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$, as the side groupoid is investigated in Section 5. The main result is given in Theorem 5.9.

In Section 6 we shortly mention some questions which naturally arise in the investigated theory and which will be the subject of subsequent papers.

In the Appendix we collect some basic notions concerning \mathcal{VB} -groupoids theory.

1 Groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of W^* -algebra

According to [17] we consider the subset $\mathcal{G}(\mathfrak{M})$ of a W^* -algebra \mathfrak{M} which consists of such elements $x \in \mathfrak{M}$ for which $|x| = (x^*x)^{\frac{1}{2}}$ is an invertible element of the W^* -subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$, where the orthogonal projection p is the support of $|x|$. We have natural maps $l : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and $r : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of $\mathcal{G}(\mathfrak{M})$ on the lattice $\mathcal{L}(\mathfrak{M})$ of orthogonal projections of the W^* -algebra \mathfrak{M} , which are defined as the left and right support of $x \in \mathcal{G}(\mathfrak{M})$, respectively. The set $\mathcal{G}(\mathfrak{M})$ possesses a structure of the groupoid where the target $\mathbf{t} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and source $\mathbf{s} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ maps are given by l and r , respectively. The partial multiplication of $x, y \in \mathcal{G}(\mathfrak{M})$ is the algebra product in \mathfrak{M} restricted to such pairs $(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M})$ for which $\mathbf{t}(y) = \mathbf{s}(x)$. The identity section $\mathbf{1} : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is defined as the inclusion map and the inversion $\iota : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is defined by

$$\iota(x) = x^{-1} := |x|^{-1}u^*, \quad (1.1)$$

where the partial isometry u and $|x|$ are uniquely defined by the polar decomposition $x = u|x|$ of $x \in \mathfrak{M}$, if one assumes that u^*u is equal to the support of $|x|$, [20].

In general the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is not a topological groupoid with respect to any natural topology on \mathfrak{M} , see [17]. However, the complex Banach manifold structures, consistent with the groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, could be defined on $\mathcal{G}(\mathfrak{M})$ and on $\mathcal{L}(\mathfrak{M})$. Therefore, following [17] for any $p \in \mathcal{L}(\mathfrak{M})$ we define the subset $\Pi_p \subset \mathcal{L}(\mathfrak{M})$ of orthogonal projections $q \in \mathcal{L}(\mathfrak{M})$ for which the Banach splitting

$$\mathfrak{M} = q\mathfrak{M} \oplus (1 - p)\mathfrak{M} \quad (1.2)$$

of \mathfrak{M} is valid. Using (1.2) we decompose

$$p = x_p - y_p \quad (1.3)$$

the projection $p \in \mathcal{L}(\mathfrak{M})$, where $x_p \in q\mathfrak{M}p$ and $y_p \in (1-p)\mathfrak{M}p$, and in this way define a bijection

$$\Pi_p \ni q \mapsto y_p =: \varphi_p(q) = (pq)^{-1} - p \in (1-p)\mathfrak{M}p, \quad (1.4)$$

and the local section

$$\Pi_p \ni q \mapsto x_p =: \sigma_p(q) = (pq)^{-1} \in \mathfrak{t}^{-1}(\Pi_p) \quad (1.5)$$

of the target map $\mathfrak{t} : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. Note here that $pq \in \mathcal{G}_q^p(\mathfrak{M}) := \mathfrak{t}^{-1}(p) \cap \mathfrak{s}^{-1}(q) \subset p\mathfrak{M}q$.

In [17] it is shown that the atlas consisting of charts (Π_p, φ_p) , where $p \in \mathcal{L}(\mathfrak{M})$, defines a complex Banach manifold structure on $\mathcal{L}(\mathfrak{M})$. The transition maps $\varphi_{p'} \circ \varphi_p^{-1} : \varphi_{p'}(\Pi_{p'} \cap \Pi_p) \rightarrow \varphi_p(\Pi_{p'} \cap \Pi_p)$ for the charts (1.4) are the following:

$$y_{p'} = (\varphi_{p'} \circ \varphi_p^{-1})(y_p) = (b + dy_p)(a + cy_p)^{-1}, \quad (1.6)$$

where $a = p'p$, $b = (1-p')p$, $c = p'(1-p)$ and $d = (1-p')(1-p)$.

The complex Banach manifold structure on $\mathcal{G}(\mathfrak{M})$ is defined by the atlas which consists of charts:

$$\begin{aligned} \Omega_{p\tilde{p}} &:= \mathfrak{t}^{-1}(\Pi_p) \cap \mathfrak{s}^{-1}(\Pi_{\tilde{p}}) \neq \emptyset \\ \psi_{p\tilde{p}} : \Omega_{p\tilde{p}} &\rightarrow (1-p)\mathfrak{M}p \oplus p\mathfrak{M}\tilde{p} \oplus (1-\tilde{p})\mathfrak{M}\tilde{p}, \end{aligned} \quad (1.7)$$

where $(p, \tilde{p}) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ and

$$\psi_{p\tilde{p}}(x) := (\varphi_p(\mathfrak{t}(x)), (\sigma_p(\mathfrak{t}(x)))^{-1}x\sigma_{\tilde{p}}(\mathfrak{s}(x)), \varphi_{\tilde{p}}(\mathfrak{s}(x))) =: (y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}). \quad (1.8)$$

We note that $z_{p\tilde{p}} \in \mathcal{G}_{\tilde{p}}^p(\mathfrak{M})$ and $\mathcal{G}_{\tilde{p}}^p(\mathfrak{M})$ is an open subset of $p\mathfrak{M}\tilde{p}$. The transition maps $\psi_{p'\tilde{p}'} \circ \psi_{p\tilde{p}}^{-1} : \psi_{p\tilde{p}}(\Omega_{p'\tilde{p}'} \cap \Omega_{p\tilde{p}}) \rightarrow \psi_{p'\tilde{p}'}(\Omega_{p'\tilde{p}'} \cap \Omega_{p\tilde{p}})$ between the above type charts are given by

$$\begin{aligned} y_{p'} &= (b + dy_p)(a + cy_p)^{-1}, \\ z_{p'\tilde{p}'} &= (a + cy_p)z_{p\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} \\ \tilde{y}_{\tilde{p}'} &= (\tilde{b} + \tilde{d}\tilde{y}_{\tilde{p}})(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1}, \end{aligned} \quad (1.9)$$

where

$$(y_{p'}, z_{p'\tilde{p}'}, \tilde{y}_{\tilde{p}'}) = (\psi_{p'\tilde{p}'} \circ \psi_{p\tilde{p}}^{-1})(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$$

and $\tilde{a} = \tilde{p}'\tilde{p}$, $\tilde{b} = (1-\tilde{p}')\tilde{p}$, $\tilde{c} = \tilde{p}'(1-\tilde{p})$ and $\tilde{d} = (1-\tilde{p}')(1-\tilde{p})$. For more details we refer to [17].

The above complex Banach manifold structures on $\mathcal{L}(\mathfrak{M})$ and on $\mathcal{G}(\mathfrak{M})$ define the corresponding underlying topological structures. Recall that the base of the underlying topology of a manifold is given by the domains of charts of the maximal atlas, see [4], §5.1.

Proposition 1.1. *The complex Banach groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ endowed with the underlying topology is a Hausdorff topological groupoid.*

Proof. Let us assume that in $\mathcal{G}(\mathfrak{M})$ there exist $x_1 \neq x_2$ such that any open neighborhoods $\Omega_1 \ni x_1$ and $\Omega_2 \ni x_2$ intersect $\Omega_1 \cap \Omega_2 \neq \emptyset$. For $i = 1, 2$ let us take $\Omega_i = \Omega_{\varepsilon_i \delta_i \tilde{\varepsilon}_i} := \psi_{p_i \tilde{p}_i}^{-1}(K_{\varepsilon_i}(0) \times K_{\delta_i}(z_{p_i \tilde{p}_i}^0) \times K_{\tilde{\varepsilon}_i}(0))$, where $p_i = \mathfrak{t}(x_i)$, $\tilde{p}_i = \mathfrak{s}(x_i)$, $K_{\varepsilon_i}(0) \subset (1-p_i)\mathfrak{M}p_i$ and $K_{\tilde{\varepsilon}_i}(0) \subset (1-\tilde{p}_i)\mathfrak{M}\tilde{p}_i$ are open balls of the radiuses ε_i and $\tilde{\varepsilon}_i$ centered at zero; also $K_{\delta_i}(z_{p_i \tilde{p}_i}^0) \subset \mathcal{G}_{\tilde{p}_i}^{p_i} \subset p_i\mathfrak{M}\tilde{p}_i$ is an open ball centered in $z_{p_i \tilde{p}_i}^0$, where $\psi_{p_i \tilde{p}_i}(x_i) = (0, z_{p_i \tilde{p}_i}^0, 0)$. If $x \in \Omega_1 \cap \Omega_2$ then from (1.9) one obtains

$$\begin{aligned} y_{p_2}(a + cy_{p_1}) &= b + dy_{p_1}, \\ z_{p_2 \tilde{p}_2}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}_1}) &= (a + cy_{p_1})z_{p_1 \tilde{p}_1} \\ \tilde{y}_{\tilde{p}_2}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}_1}) &= \tilde{b} + \tilde{d}\tilde{y}_{\tilde{p}_1}, \end{aligned} \quad (1.10)$$

where

$$\psi_{p_2\tilde{p}_2}^{-1}(y_{p_2}, z_{p_2\tilde{p}_2}, \tilde{y}_{\tilde{p}_2}) = \psi_{p_1\tilde{p}_1}^{-1}(y_{p_1}, z_{p_1\tilde{p}_1}, \tilde{y}_{\tilde{p}_1}) = x.$$

The other quantities in (1.10) are given by $a = p_2p_1$, $b = (1 - p_2)p_1$, $c = p_2(1 - p_1)$, $d = (1 - p_2)(1 - p_1)$, $\tilde{a} = \tilde{p}_2\tilde{p}_1$, $\tilde{b} = (1 - \tilde{p}_2)\tilde{p}_1$, $\tilde{c} = \tilde{p}_2(1 - \tilde{p}_1)$ and $\tilde{d} = (1 - \tilde{p}_2)(1 - \tilde{p}_1)$. The elements $x \in \Omega_{\varepsilon_1\delta_1\tilde{\varepsilon}_1} \cap \Omega_{\varepsilon_2\delta_2\tilde{\varepsilon}_2}$ in the limit $\varepsilon_1, \delta_1, \tilde{\varepsilon}_1, \varepsilon_2, \delta_2, \tilde{\varepsilon}_2 \rightarrow 0$ go to $x \rightarrow x_1$ and $x \rightarrow x_2$. Thus and from (1.10) we have

$$\begin{aligned} 0 &= b \\ z_{p_2\tilde{p}_2}^0 \tilde{a} &= az_{p_1\tilde{p}_1}^0 \\ 0 &= \tilde{b} \end{aligned} \tag{1.11}$$

The above gives

$$\begin{aligned} 0 &= p_1p_2 - p_2 \\ z_{p_2\tilde{p}_2}^0 &= p_2z_{p_1\tilde{p}_1}^0\tilde{p}_2 \\ 0 &= \tilde{p}_1\tilde{p}_2 - \tilde{p}_2 \end{aligned} \tag{1.12}$$

Taking in (1.9) the transition map $\psi_{p_1\tilde{p}_1} \circ \psi_{p_2\tilde{p}_2}^{-1}$ instead of $\psi_{p_2\tilde{p}_2} \circ \psi_{p_1\tilde{p}_1}^{-1}$ and repeating the analogous considerations we obtain

$$\begin{aligned} 0 &= p_2p_1 - p_1 \\ z_{p_1\tilde{p}_1}^0 &= p_1z_{p_2\tilde{p}_2}^0\tilde{p}_1 \\ 0 &= \tilde{p}_2\tilde{p}_1 - \tilde{p}_1 \end{aligned} \tag{1.13}$$

From (1.12) and (1.13) we find that $p_1 = p_2$, $\tilde{p}_1 = \tilde{p}_2$ and $z_{p_1\tilde{p}_1}^0 = z_{p_2\tilde{p}_2}^0$. It means that $x_1 = x_2$ which is in the contradiction to the assumption that $x_1 \neq x_2$. Thus we conclude that the topology underlying the complex Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ is Hausdorff one. Since $\mathcal{L}(\mathfrak{M})$ is a submanifold of $\mathcal{G}(\mathfrak{M})$ the underlying topology of $\mathcal{L}(\mathfrak{M})$ is also a Hausdorff one. \square

Keeping in mind Proposition 1.1 and the definition of Lie groupoid in the finite dimensional case (see e.g. [9, 13]) we will consider $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a Banach-Lie groupoid.

Let us fix $p_0 \in \mathcal{L}(\mathfrak{M})$ and define $\mathcal{L}_{p_0}(\mathfrak{M}) := \{p \in \mathcal{L}(\mathfrak{M}) : p \sim p_0\}$ and $\mathcal{G}_{p_0}(\mathfrak{M}) := \mathfrak{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap \mathfrak{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M}))$, where $p \sim p'$ denotes the Murray - von Neumann equivalence of projections. Then $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is a Banach-Lie subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ defined unambiguously by the choice of $p_0 \in \mathcal{L}(\mathfrak{M})$. The total space $\mathcal{G}_{p_0}(\mathfrak{M})$ as well as the base $\mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ are subsets of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{L}(\mathfrak{M})$, respectively, open with respect to the topology defined by their Banach manifold structures. If $p \sim p_0$ then the groupoid $\mathcal{G}_p(\mathfrak{M}) \rightrightarrows \mathcal{L}_p(\mathfrak{M})$ coincides with $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$. If $\Pi_p \cap \mathcal{L}_{p_0}(\mathfrak{M}) \neq \emptyset$ then for $q \in \Pi_p \cap \mathcal{L}_{p_0}(\mathfrak{M})$ one has $q \sim p$ and $q \sim p_0$. Thus one obtains $p \sim p_0$, and so $\Pi_p \subset \mathcal{L}_{p_0}(\mathfrak{M})$. Hence, for $p \not\sim p_0$ one has $\Pi_p \cap \mathcal{L}_{p_0}(\mathfrak{M}) = \emptyset$. From the above it follows that, for $p \in \mathcal{L}(\mathfrak{M})$, domain of chart $\varphi_p : \Pi_p \rightarrow (1 - p)\mathfrak{M}p$ is contained in $\mathcal{L}_{p_0}(\mathfrak{M})$ if and only if $p \sim p_0$. As a conclusion we have

Proposition 1.2. *The Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a disjoint union of Banach-Lie subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$, which are its closed-open Banach subgroupoids.*

Let us denote by $C(\mathfrak{M})$ the center of \mathfrak{M} . The next proposition characterizes the set $C(\mathfrak{M}) \cap \mathcal{L}(\mathfrak{M})$ of central projections of the W^* -algebra \mathfrak{M} in the terms of the underlying topology of the complex Banach manifold structure of $\mathcal{L}(\mathfrak{M})$.

Proposition 1.3. *One has the following statements:*

(i) $p \in C(\mathfrak{M}) \cap \mathcal{L}(\mathfrak{M})$ if and only if $\Pi_p = \{p\}$;

(ii) If $p \notin C(\mathfrak{M}) \cap \mathcal{L}(\mathfrak{M})$ then $\Pi_p \cap C(\mathfrak{M}) = \emptyset$.

Proof. (i) If p is a central projection then $(1-p)\mathfrak{M}p = 0$. So, for $q \in \Pi_p$ one has $\varphi_p(q) = y_p = 0$, and thus $\Pi_p = \{p\}$.

Let p be a projection such that $\Pi_p = \{p\}$. This means that

$$(1-p)\mathfrak{M}p = 0, \quad (1.14)$$

because $\varphi_p : \Pi_p \rightarrow (1-p)\mathfrak{M}p$ is a bijection. According to Lemma 1.7 of Chapter V in [21] the condition (1.14) is equivalent to

$$c(1-p)c(p) = 0, \quad (1.15)$$

where $c(1-p)$ and $c(p)$ are central supports of $1-p$ and p , respectively. Since $c(1-p) \geq 1-p$ and $c(p) \geq p$ from (1.15) one has

$$c(1-p) + c(p) = (c(1-p) + c(p))(1-p+p) = 1-p+p = 1. \quad (1.16)$$

From (1.15) and (1.16) follows that $c(1-p) = 1-p$ and $c(p) = p$. So p is a central projection.

(ii) Let us assume that $\Pi_p \cap C(\mathfrak{M}) \neq \emptyset$ and take $q \in \Pi_p \cap C(\mathfrak{M})$. It follows from (i) of this proposition that $\Pi_q = \{q\}$ and, thus $\varphi_q(q) = 0$. From (1.6) one obtains

$$\varphi_p(q) = (\varphi_p \circ \varphi_q^{-1})(0) = ba^{-1}, \quad (1.17)$$

where $a = pq$ and $b = (1-p)q$. Since q is a central projection the elements a and b are also projections. So, $a^{-1} = a$, and thus $ba = 0$. The above shows that $\varphi_p(q) = 0$, i.e. $p = q \in C(\mathfrak{M}) \cap \mathcal{L}(\mathfrak{M})$. This is in the contradiction with $p \notin \Pi_p \cap C(\mathfrak{M})$. \square

We conclude from Proposition 1.3 that $C(\mathfrak{M}) \cap \mathcal{L}(\mathfrak{M})$ coincides with the set of elements of $\mathcal{L}(\mathfrak{M})$ which are open-closed one element subsets of $\mathcal{L}(\mathfrak{M})$. In the case when \mathfrak{M} is a commutative W^* -algebra one has $\Pi_p = \{p\}$ for any $p \in \mathcal{L}(\mathfrak{M})$. Therefore, for such \mathfrak{M} the Banach manifold structure of $\mathcal{L}(\mathfrak{M})$ defined by the atlas (1.5) is trivial, i.e. $\mathcal{L}(\mathfrak{M})$ is 0-dimensional manifold.

Let us recall here that the inner subgroupoid $\mathcal{J} \rightrightarrows B$ of a groupoid $G \rightrightarrows B$ is defined as $\mathcal{J} := \bigcup_{b \in B} \mathbf{s}^{-1}(b) \cap \mathbf{t}^{-1}(b)$. The inner subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ we will denote by $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. For a commutative algebra \mathfrak{M} one has $l(x) = xx^{-1} = x^{-1}x = r(x)$. So, the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ coincides with its inner subgroupoid $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Summing up the facts mentioned above we see that in the case of a commutative W^* -algebra \mathfrak{M} one can consider $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as the disjoint union of Banach-Lie groups $G(p\mathfrak{M}p)$ enumerated by $p \in \mathcal{L}(\mathfrak{M})$.

From Proposition 1.2 it follows that in order to study the Banach-Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ it is enough to investigate the structure of Banach-Lie subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$. For this reason let us note that the source map $\mathbf{s} : \mathcal{G}_{p_0}(\mathfrak{M}) \rightarrow \mathcal{L}_{p_0}(\mathfrak{M})$ is a surjective submersion which in the coordinates (1.8) assumes the form $(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \mapsto \tilde{y}_{\tilde{p}}$. So, the fibre $P_0 := \mathbf{s}^{-1}(p_0)$ of the source map $\mathbf{s} : \mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is a Banach submanifold of $\mathcal{G}_{p_0}(\mathfrak{M})$. Restricting (1.7) and (1.8) to $P_0 \cap \Omega_{pp_0} = \mathbf{t}^{-1}(\Pi_p) \cap \mathbf{s}^{-1}(p_0) = \pi_0^{-1}(\Pi_p)$, where $\pi_0 := \mathbf{t}|_{P_0}$, one obtains the charts

$$\psi_p : \pi_0^{-1}(\Pi_p) \xrightarrow{\sim} (1-p)\mathfrak{M}p \times \mathcal{G}_{p_0}^p(\mathfrak{M}) \quad (1.18)$$

which define the atlas $(\pi_0^{-1}(\Pi_p), \psi_p)$, $p \in \mathcal{L}_{p_0}(\mathfrak{M})$, on P_0 . For $y_p \in (1-p)\mathfrak{M}p$, $z_{pp_0} \in \mathcal{G}_{p_0}^p(\mathfrak{M})$ and $\eta \in \pi_0^{-1}(\Pi_p)$ one has

$$\eta = \psi_p^{-1}(y_p, z_{pp_0}) = (p + y_p)z_{pp_0} \quad (1.19)$$

and $\psi_p(\eta) = (y_p, z_{pp_0})$, where

$$\begin{aligned} y_p &= \eta(p\eta)^{-1} - p \\ z_{pp_0} &= p\eta \end{aligned} \quad (1.20)$$

Let us note that P_0 is an open subset of the Banach space $\mathfrak{M}p_0$ and, thus one can consider (P_0, id) as a chart on P_0 . The transition maps (1.19) and (1.20) show that (P_0, id) belongs to the maximal atlas of the manifold P_0 defined by $(\pi_0^{-1}(\Pi_p), \psi_p)$, $p \in \mathcal{L}_{p_0}(\mathfrak{M})$. Hence we conclude that the topologies of P_0 inherited from $\mathcal{G}_{p_0}(\mathfrak{M})$ and from $\mathfrak{M}p_0$ are the same. The free right actions of the Banach-Lie group $G_0 := G(p_0\mathfrak{M}p_0)$ of the invertible elements of the W^* -subalgebra $p_0\mathfrak{M}p_0$ on P_0 and on $P_0 \times P_0$ are defined by

$$\kappa : P_0 \times G_0 \ni (\eta, g) \mapsto \eta g \in P_0 \quad (1.21)$$

and by

$$\kappa_2 : P_0 \times P_0 \times G_0 \ni (\eta, \xi, g) \mapsto (\eta g, \xi g) \in P_0 \times P_0, \quad (1.22)$$

respectively. They are consistent with the atlas $(\pi_0^{-1}(\Pi_p), \psi_p)$, $p \in \mathcal{L}(\mathfrak{M})$ defined in (1.18), and the atlas $(\pi_0^{-1}(\Pi_p) \times \pi_0^{-1}(\Pi_{\tilde{p}}), \psi_p \times \psi_{\tilde{p}})$, $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$, defined by

$$(\pi_0^{-1}(\Pi_p) \times \pi_0^{-1}(\Pi_{\tilde{p}})) \ni (\eta, \xi) \mapsto (y_p, z_{pp_0}, y_{\tilde{p}}, z_{\tilde{p}p_0}), \quad (1.23)$$

i.e. one has $\psi_p(\eta g) = (y_p, z_{pp_0}g)$ and $(\psi_p \times \psi_q)(\eta g, \xi g) = (y_p, z_{pp_0}g, y_q, z_{qq_0}g)$. The orbits of G_0 on P_0 and $P_0 \times P_0$ coincide with the fibres of the submersions

$$\varphi : P_0 \ni \eta \mapsto \varphi(\eta) := \eta\eta^{-1} \in \mathcal{L}_{p_0}(\mathfrak{M}), \quad (1.24)$$

$$\phi : P_0 \times P_0 \ni (\eta, \xi) \mapsto \phi(\eta, \xi) := \eta\xi^{-1} \in \mathcal{G}_{p_0}(\mathfrak{M}). \quad (1.25)$$

So, the equivalence relations defined by the actions (1.21) and (1.22) are regular in sense of the definition given in 5.9.5 of [4]. Thus the quotient spaces P_0/G_0 and $\frac{P_0 \times P_0}{G_0}$ are Banach manifolds isomorphic to $\mathcal{L}_{p_0}(\mathfrak{M})$ and $\mathcal{G}_{p_0}(\mathfrak{M})$, respectively.

Let us consider the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$ and the action groupoid $P_0 \rtimes G_0 \rightrightarrows P_0$ which are, as one can easily see, a Banach-Lie groupoids. The definition of the action groupoid one can find in [13]. We define $\iota : P_0 \rtimes G_0 \rightarrow P_0 \times P_0$ by

$$\iota(\eta, g) := (\eta g, \eta). \quad (1.26)$$

Proposition 1.4. (i) One has the following (non exact) sequence of groupoid morphisms

$$\begin{array}{ccccc} P_0 \rtimes G_0 & \xrightarrow{\iota} & P_0 \times P_0 & \xrightarrow{\phi} & \mathcal{G}_{p_0}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ P_0 & \xrightarrow{id} & P_0 & \xrightarrow{\varphi} & \mathcal{L}_{p_0}(\mathfrak{M}). \end{array} \quad (1.27)$$

where the pairs of maps (ι, id) and (ϕ, φ) define groupoid monomorphism and epimorphism, respectively.

(ii) The quotient groupoid (gauge groupoid) $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ and the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ are isomorphic, where the isomorphism

$$\begin{array}{ccc} \frac{P_0 \times P_0}{G_0} & \xrightarrow{[\phi]} & \mathcal{G}_{p_0}(\mathfrak{M}) \\ \Downarrow & & \Downarrow \\ P_0/G_0 & \xrightarrow{[\varphi]} & \mathcal{L}_{p_0}(\mathfrak{M}) \end{array}, \quad (1.28)$$

is given by the quotienting of (1.24) and (1.25).

(iii) The quotient groupoid $\frac{P_0 \rtimes G_0}{G_0} \rightrightarrows P_0/G_0$ of $P_0 \rtimes G_0 \rightrightarrows P_0$ is isomorphic to the inner subgroupoid $\mathcal{J}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$.

All groupoid morphisms mentioned above are morphisms of Banach-Lie groupoids.

Proof. Straightforward after observation that all arrows in (1.27) are given by G_0 -equivariant maps. \square

We note that $\varphi = \pi_0 : P_0 \rightarrow \mathcal{L}_{p_0}(\mathfrak{M})$ and $\phi : P_0 \times P_0 \rightarrow \mathcal{G}_{p_0}(\mathfrak{M})$ are the projections on bases of the G_0 -principal bundles.

The map $\phi : P_0 \times P_0 \ni (\eta, \xi) \mapsto \eta\xi^{-1} = x \in \mathcal{G}_{p_0}(\mathfrak{M})$ written in the coordinates (1.8) and coordinates (1.23) assumes the form

$$(y_p, z_{pp_0}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0}) \mapsto (y_p, z_{pp_0} \tilde{z}_{\tilde{p}p_0}^{-1}, \tilde{y}_{\tilde{p}}) = (y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}). \quad (1.29)$$

In the subsequent considerations we will identify $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ with the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$.

2 Atiyah sequence of the principal bundle $P_0 \rightarrow P_0/G_0$

Following of [18] we describe the Atiyah sequence of the principal bundle $P_0 \rightarrow P_0/G_0$ as well as the principal bundle $P_0 \times P_0 \rightarrow \frac{P_0 \times P_0}{G_0}$. Isomorphic realizations of the tangent groupoid $T\mathcal{G}_{p_0} \rightrightarrows T\mathcal{L}_{p_0}(\mathfrak{M})$ will be presented in Proposition 2.1. The atlases on TP_0 and on $T\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows T\mathcal{L}_{p_0}(\mathfrak{M})$ consistent with their bundle structures will be also described.

We start from the description of the tangent group TG_0 of G_0 and the tangent bundle TP_0 of P_0 . We will identify TG_0 with the semidirect product $p_0\mathfrak{M}p_0 \rtimes_{Ad_{G_0}} G_0$ of G_0 with its Lie algebra $T_e G_0 \cong p_0\mathfrak{M}p_0$ by

$$TG_0 \ni X_g \mapsto (TR_{g^{-1}}(g)X_g, g) =: (x, g) \in p_0\mathfrak{M}p_0 \rtimes_{Ad_{G_0}} G_0. \quad (2.1)$$

Thus the group product

$$X_g \bullet Y_h = TL_g(h)Y_h + TR_h(g)X_g$$

of $X_g \in T_g G_0$ and $Y_h \in T_h G_0$ can be written as

$$(x, g) \bullet (y, h) = (x + gyg^{-1}, gh), \quad (2.2)$$

where $x = TR_{g^{-1}}(g)X_g$ and $y = TR_{h^{-1}}(h)Y_h$. One has the short exact sequence

$$\{e\} \rightarrow T_e G_0 \rightarrow TG_0 \rightarrow G_0 \rightarrow \{e\} \quad (2.3)$$

of groups which is isomorphic to the following one

$$\{e\} \rightarrow p_o \mathfrak{M}p_0 \rightarrow p_0 \mathfrak{M}p_0 \rtimes_{Ad_{G_0}} G_0 \rightarrow G_0 \rightarrow \{e\}. \quad (2.4)$$

We note that $T_e G_0 \cong p_0 \mathfrak{M}p_0$ and G_0 are subgroups of TG_0 .

The inclusion map $\iota : P_0 \hookrightarrow \mathfrak{M}p_0$ maps P_0 on the open subset of the Banach space $\mathfrak{M}p_0$. So, its tangent map $T\iota : TP_0 \xrightarrow{\sim} \mathfrak{M}p_0 \times P_0$ defines a chart on TP_0 with $(v, \eta) \in \mathfrak{M}p_0 \times P_0$ as the coordinates of an element of TP_0 .

The actions (1.21) and (1.22) of G_0 on P_0 and on $P_0 \times P_0$ define the corresponding actions of TG_0 on TP_0 and on $T(P_0 \times P_0)$ which are given by

$$T\kappa : TP_0 \times TG_0 \ni ((\vartheta, \eta), (x, g)) \mapsto (\vartheta g + \eta x g, \eta g) \in TP_0, \quad (2.5)$$

and by

$$T\kappa_2 : (TP_0 \times TP_0) \times TG_0 \ni ((\vartheta, \eta), (\omega, \xi), (x, g)) \mapsto ((\vartheta g + \eta x g, \eta g), (\omega g + \xi x g, \xi g)) \in TP_0 \times TP_0. \quad (2.6)$$

where $(\vartheta, \eta), (\omega, \xi) \in \mathfrak{M}p_0 \times P_0$ and $(x, g) \in p_0 \mathfrak{M}p_0 \rtimes_{Ad_{G_0}} G_0$. Orbits of the normal subgroup $p_0 \mathfrak{M}p_0 \cong T_e G_0 \subset TG_0$ are the affine subspaces $\{(v + \eta x, \eta) : x \in p_0 \mathfrak{M}p_0\} \subset T_\eta P_0$ of the tangent space $T_\eta P_0$ at $\eta \in P_0$. Thus the vertical tangent subspace $T^V P_0 \subset TP_0$ consists of the orbits generated from $(0, \eta) \in TP_0$, i.e. $T_\eta^V P_0 = \{(\eta x, \eta) : x \in p_0 \mathfrak{M}p_0\}$.

It follows from (1.2) that for any $\eta \in \pi_0^{-1}(\Pi_p)$ one has the Banach splitting

$$T_\eta P_0 \cong (q \mathfrak{M}p_0 \oplus (1 - p) \mathfrak{M}p_0) \times \{\eta\} \quad (2.7)$$

of the tangent space $T_\eta P_0 \cong \mathfrak{M}p_0 \times \{\eta\}$, where $q = \eta \eta^{-1}$, and $T_\eta^V P_0 \cong q \mathfrak{M}p_0 \times \{\eta\} = \eta \mathfrak{M}p_0 \times \{\eta\}$. Using (2.7) we decompose $(v, \eta) \in T(\pi_0^{-1}(\Pi_p))$ on $(v^V(q) + v^h(q), \eta)$ where $v^V(q) \in q \mathfrak{M}p_0$ and $v^h(q) \in (1 - p) \mathfrak{M}p_0$, and obtain in this way the local trivialization

$$\tau_p : T(\pi_0^{-1}(\Pi_p)) \ni (v, \eta) \mapsto (\sigma_p^{-1}(q) v^V(q) + v^h(q), \eta) \in (p \mathfrak{M}p_0 \oplus (1 - p) \mathfrak{M}p_0) \times \pi_0^{-1}(\Pi_p) \quad (2.8)$$

of TP_0 which satisfies

$$\tau_p(T(\pi_0^{-1}(\Pi_p)) \cap T^V P_0) = p \mathfrak{M}p_0 \times \pi_0^{-1}(\Pi_p). \quad (2.9)$$

Hence according to 7.5.1 in [4] the vertical bundle $T^V P_0$ is a Banach vector subbundle of TP_0 . So, from 7.5.2 of [4] it follows that the quotient bundle $TP_0/T^V P_0 \cong TP_0/T_e G_0$ is a Banach vector bundle over P_0 . The above facts one summarizes as the short exact sequence

$$T^V P_0 \longrightarrow TP_0 \longrightarrow TP_0/T_e G_0 \quad (2.10)$$

of the Banach vector bundles over P_0 .

In the following we will identify $p_0 \mathfrak{M}p_0 \times P_0$ with $T^V P_0$ by the vector bundle isomorphism

$$I : p_0 \mathfrak{M}p_0 \times P_0 \ni (x, \eta) \mapsto I(x, \eta) := (\eta x, \eta) \in T^V P_0. \quad (2.11)$$

Let us note here that the quotient map $A : TP_0 \rightarrow TP_0/T_e G_0$ is defined as follows

$$\mathfrak{M}p_0 \times P_0 \ni (v, \eta) \mapsto A(v, \eta) := \{(v + \eta x, \eta) : x \in p_0 \mathfrak{M}p_0\}. \quad (2.12)$$

The action (2.5) restricted to the subgroup $G_0 \subset TG_0$ is regular and preserves the structure of (2.10). So, quotienting (2.10) by G_0 one obtains the short exact sequence of the Banach vector

bundles

$$\begin{array}{ccccc}
p_0\mathfrak{M}_{p_0} \times_{Ad_{G_0}} P_0 & \xrightarrow{\iota} & TP_0/G_0 & \xrightarrow{a} & T(P_0/G_0) \\
\downarrow & & \downarrow & & \downarrow \\
P_0/G_0 & \xrightarrow{\sim} & P_0/G_0 & \xrightarrow{\sim} & P_0/G_0
\end{array} \tag{2.13}$$

over P_0/G_0 , which is the Atiyah sequence of the principal bundle $\pi_0 : P_0 \rightarrow P_0/G_0$. One can find the definition of Atiyah sequence of a principal bundle, e.g. in [1, 13]. In order to obtain (2.13) we have used the bundles morphisms

$$\iota := [I] : p_0\mathfrak{M}_{p_0} \times_{Ad_{G_0}} P_0 \rightarrow T^V P_0/G_0, \tag{2.14}$$

$$a := [A] : TP_0/G_0 \rightarrow (TP_0/T_e G_0)/G_0 \cong TP_0/TG_0 \cong T(P_0/G_0) \tag{2.15}$$

which follow from (2.5).

The same argumentation applied to the action of G_0 on $P_0 \times P_0$ defined in (2.6) leads to the Atiyah sequence

$$\begin{array}{ccccc}
p_0\mathfrak{M}_{p_0} \times_{Ad_{G_0}} (P_0 \times P_0) & \xrightarrow{\iota_2} & \frac{T(P_0 \times P_0)}{G_0} & \xrightarrow{a_2} & T\left(\frac{P_0 \times P_0}{G_0}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0}
\end{array} \tag{2.16}$$

of the G_0 -principal bundle $\pi_{02} : P_0 \times P_0 \rightarrow \frac{P_0 \times P_0}{G_0}$, where ι_2 and a_2 are defined by the quotienting of

$$I_2 : p_0\mathfrak{M}_{p_0} \times P_0 \times P_0 \ni (x, \eta, \xi) \mapsto (\eta x, \eta, \xi x, \xi) \in TP_0 \times TP_0 \tag{2.17}$$

and

$$A_2 : TP_0 \times TP_0 \ni (v, \eta, w, \xi) \mapsto \{(v + \eta x, \eta, w + \xi x, \xi); x \in p_0\mathfrak{M}_{p_0}\} \in \frac{TP_0 \times TP_0}{T_e G_0}, \tag{2.18}$$

respectively.

The proposition given below presents the equivalent representations of the tangent groupoid $T\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows T\mathcal{L}_{p_0}(\mathfrak{M})$.

Proposition 2.1. *One has the following groupoid isomorphisms*

$$\begin{array}{ccccc}
\frac{TP_0 \times TP_0}{TG_0} & \xrightarrow{\Delta} & T\left(\frac{P_0 \times P_0}{G_0}\right) & \xrightarrow{T[\phi]} & T\mathcal{G}_{p_0}(\mathfrak{M}) \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
\frac{TP_0}{TG_0} & \xrightarrow{\delta} & T\left(\frac{P_0}{G_0}\right) & \xrightarrow{T[\varphi]} & T\mathcal{L}_{p_0}(\mathfrak{M}).
\end{array} \tag{2.19}$$

Proof. One has the canonically defined isomorphism between the tangent groupoid $T(P_0 \times P_0) \rightrightarrows TP_0$ of the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$ and the pair groupoid $TP_0 \times TP_0 \rightrightarrows TP_0$. Therefore using vector bundles isomorphisms

$$\delta : TP_0/TG_0 \xrightarrow{\sim} T(P_0/G_0), \quad (2.20)$$

$$\Delta : \frac{TP_0 \times TP_0}{TG_0} \xrightarrow{\sim} T\left(\frac{P_0 \times P_0}{G_0}\right) \quad (2.21)$$

defined by the actions (2.5) and (2.6) and applying the tangent functor to the groupoid isomorphism given in (1.28) we obtain the isomorphisms mentioned in (2.19). \square

Now we define the atlas on TP_0 consistent with the principal bundle structure of P_0 . To this end we will use the charts (1.18) defined by (1.19). In order to find the explicit formula for the chart $T\psi_p = T(\pi_0^{-1}(\Pi_p)) \rightarrow (1-p)\mathfrak{M}p \times p\mathfrak{M}p_0 \times (1-p)\mathfrak{M}p \times \mathcal{G}_{p_0}^p(\mathfrak{M})$ tangent to ψ_p we consider a smooth curve

$$]-\varepsilon, \varepsilon[\ni t \mapsto \eta(t) = (p + y_p(t))z_{pp_0}(t) \in \pi_0^{-1}(\Pi_p), \quad (2.22)$$

such that $\eta(0) = \eta \in P_0$. Differentiating (2.22) with respect the parameter t at $t = 0$ and defining

$$a_p := \frac{d}{dt}y_p(t)|_{t=0} \in (1-p)\mathfrak{M}p \quad (2.23)$$

$$b_p := \frac{d}{dt}z_{pp_0}(t)|_{t=0}z_{pp_0}^{-1} \in p\mathfrak{M}p \quad (2.24)$$

we obtain

$$v = (a_p + (p + y_p)b_p)z_{pp_0}. \quad (2.25)$$

Let us note here that equalities $\eta(0) = \eta$ and

$$v = \frac{d}{dt}\eta(t)|_{t=0} \in \mathfrak{M}p_0 \quad (2.26)$$

define the bundle isomorphism $TP_0 \cong \mathfrak{M}p_0 \times P_0$.

By simple calculations we can invert formulas (1.19) and (2.25) obtaining

$$a_p = (v - \eta(p\eta)^{-1}v)(p\eta)^{-1}, \quad (2.27)$$

$$b_p = pv(p\eta)^{-1}. \quad (2.28)$$

Therefore the formulas (2.27) and (2.28) taken together with (1.20) define the chart

$$T(\pi_0^{-1}(\Pi_p)) \ni (v, \eta) \mapsto T\psi_p(v, \eta) = (a_p, b_p, y_p, z_{pp_0}) \quad (2.29)$$

tangent to the chart $(\pi_0^{-1}(\Pi_p), \psi_p)$ defined in (1.20).

Note here that in (2.25) we used b_p instead of $\frac{d}{dt}z_{pp_0}(t)|_{t=0}$. Note also that the product $p\eta$ of p and η is not the product in sense of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ multiplication. However, $p\eta \in \mathcal{G}(\mathfrak{M})$ and thus one can take its groupoid inverse $(p\eta)^{-1}$.

In order to find the transition map $(T\psi_{p'} \circ T\psi_p^{-1}) : (a_p, b_p, y_p, z_{pp_0}) \mapsto (a_{p'}, b_{p'}, y_{p'}, z_{p'p_0})$ we will use for $(v, \eta) \in (T(\pi_0^{-1}(\Pi_p \cap \Pi_{p'})))$ the equalities:

$$\eta = (p + y_p)z_{pp_0} = (p' + y_{p'})z_{p'p_0} \quad (2.30)$$

$$v = [a_p + (p + y_p)b_p]z_{pp_0} = [a_{p'} + (p' + y_{p'})b_{p'}]z_{p'p_0} \quad (2.31)$$

which follow from (1.19) and (2.25), respectively. Solving equations (2.30) and (2.31) with respect to $(a_{p'}, b_{p'}, y_{p'}, z_{p'p_0})$ we obtain:

$$a_{p'} = (d - (b + dy_p)(a + cy_p)^{-1})a_p(a + cy_p)^{-1} \quad (2.32)$$

$$b_{p'} = (ca_p + (a + cy_p)b_p)(a + cy_p)^{-1} \quad (2.33)$$

$$y_{p'} = (b + dy_p)(a + cy_p)^{-1} \quad (2.34)$$

$$z_{p'p_0} = (a + cy_p)z_{pp_0}. \quad (2.35)$$

The chart (2.29) is equivariant with respect to the action of G_0 on TP_0 defined as the restriction of (2.5) to the subgroup $G_0 \subset TG_0$ and the action on $(1-p)\mathfrak{M}p \times p\mathfrak{M}p \times (1-p)\mathfrak{M}p \times p\mathfrak{M}p_0$ is defined for $g \in G_0$ by $(a_p, b_p, y_p, z_{pp_0}) \mapsto (a_p, b_p, y_p, z_{pp_0}g)$. So, after quotienting by G_0 one defines the chart

$$T(\pi_0^{-1}(\Pi_p))/G_0 \ni \langle v, \eta \rangle \mapsto [T\psi_p](\langle v, \eta \rangle) = (a_p, b_p, y_p). \quad (2.36)$$

The transition map $[T\psi_{p'}] \circ [T\psi_p]^{-1} : (a_p, b_p, y_p) \mapsto (a_{p'}, b_{p'}, y_{p'})$ between these charts is given by (2.32-2.34).

We end this section describing the atlas on $T\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows T\mathcal{L}_{p_0}(\mathfrak{M})$ consistent with the one defined by (1.7) and (1.8). So, additionally to (2.22) we define

$$\tilde{a}_{\tilde{p}} := \frac{d}{dt}\tilde{y}_{\tilde{p}}(t)|_{t=0} \in (1-\tilde{p})\mathfrak{M}\tilde{p}, \quad (2.37)$$

$$b_{p\tilde{p}} := \frac{d}{dt}z_{p\tilde{p}}(t)|_{t=0} \in p\mathfrak{M}\tilde{p}, \quad (2.38)$$

where $]-\varepsilon, \varepsilon[\ni t \mapsto x(t) = (p + y_p(t))z_{p\tilde{p}}(t)(\tilde{p} + \tilde{y}_{\tilde{p}}(t))^{-1} \in \Omega_{p\tilde{p}}$. Hence we obtain the atlas on $T\mathcal{G}_{p_0}(\mathfrak{M})$ given by the coordinates $(a_p, b_{p\tilde{p}}, \tilde{a}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$ tangent to $(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$, enumerated by $(p, \tilde{p}) \in \mathcal{L}_{p_0}(\mathfrak{M}) \times \mathcal{L}_{p_0}(\mathfrak{M})$. Using the tangent map of the transition map presented in (1.9) one obtains the map

$$\begin{aligned} a_{p'} &= da_p(a + cy_p)^{-1} - (b + dy_p)(a + cy_p)^{-1}ca_p(a + cy_p)^{-1} \\ b_{p'\tilde{p}'} &= ca_pz_{p\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} + (a + cy_p)b_{p\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} - (a + cy_p)z_{p\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1}\tilde{c}\tilde{a}_{\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} \\ \tilde{a}_{\tilde{p}'} &= \tilde{d}\tilde{a}_{\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} - (\tilde{b} + \tilde{d}\tilde{y}_{\tilde{p}})(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1}\tilde{c}\tilde{a}_{\tilde{p}}(\tilde{a} + \tilde{c}\tilde{y}_{\tilde{p}})^{-1} \end{aligned} \quad (2.39)$$

which together with (1.9) gives the transition map from the coordinates $(a_p, b_{p\tilde{p}}, \tilde{a}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$ to the coordinates $(a_{p'}, b_{p'\tilde{p}'}, \tilde{a}_{\tilde{p}'}, y_{p'}, z_{p'\tilde{p}'}, \tilde{y}_{\tilde{p}'})$. The above formulas will be used in the sequel for the coordinate expressions of the various structures and dependences between of them.

3 Atiyah sequence of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

In this section we discuss some questions which arise in a natural way when one considers the algebroid $\mathcal{AG}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of a W^* -algebra \mathfrak{M} .

We start by defining the following Banach vectors bundles over the lattice $\mathcal{L}(\mathfrak{M})$ of the orthogonal projectors of \mathfrak{M} :

$$\mathcal{A}(\mathfrak{M}) := \{(x, q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : x \in q\mathfrak{M}q\} \quad (3.1)$$

$$\mathcal{M}^L(\mathfrak{M}) := \{(x, q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : x \in \mathfrak{M}q\} \quad (3.2)$$

$$\mathcal{T}(\mathfrak{M}) := \{(x, q) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : x \in (1-q)\mathfrak{M}q\}. \quad (3.3)$$

From the Banach splitting

$$\mathfrak{M}q = q\mathfrak{M}q \oplus (1 - q)\mathfrak{M}q \quad (3.4)$$

of the left ideal $\mathfrak{M}q$ of \mathfrak{M} taken for any $q \in \mathcal{L}(\mathfrak{M})$ we find that these bundles form the following short exact sequence

$$\begin{array}{ccccc} \mathcal{A}(\mathfrak{M}) & \xrightarrow{\iota} & \mathcal{M}^L(\mathfrak{M}) & \xrightarrow{a} & \mathcal{T}(\mathfrak{M}) \\ \downarrow \pi_{\mathcal{A}} & & \downarrow \pi_{\mathcal{M}} & & \downarrow \pi_{\mathcal{T}} \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}), \end{array} \quad (3.5)$$

where the bundle monomorphism $\iota : \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{M}^L(\mathfrak{M})$ and the bundle epimorphism $a : \mathcal{M}^L(\mathfrak{M}) \rightarrow \mathcal{T}(\mathfrak{M})$ are defined by the inclusions $q\mathfrak{M}q \hookrightarrow \mathfrak{M}q$ and the projections $\mathfrak{M}q \rightarrow (1 - q)\mathfrak{M}q$ of fibres given by the splitting (3.4). All projections on the base in (3.5) are defined as the projections of the product $\mathfrak{M} \times \mathcal{L}(\mathfrak{M})$ on its second component.

One has the short exact sequence of Banach-Lie groupoids

$$\begin{array}{ccccc} \mathcal{J}(\mathfrak{M}) & \xrightarrow{\hookrightarrow} & \mathcal{G}(\mathfrak{M}) & \xrightarrow{(t,s)} & \mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) \\ \Downarrow & & \Downarrow & & \Downarrow \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}), \end{array} \quad (3.6)$$

where $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is the inner subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ defined as usually by

$$\mathcal{J}(\mathfrak{M}) := \bigcup_{q \in \mathcal{L}(\mathfrak{M})} (s^{-1}(q) \cap t^{-1}(q)). \quad (3.7)$$

The groupoid $\mathcal{L}(\mathfrak{M}) \times_{\mathcal{R}} \mathcal{L}(\mathfrak{M}) := \{(q, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) : q \sim p\}$ is a subgroupoid of the pair groupoid $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. Recall that the equivalence relation $q \sim p$ is the Murray-von Neumann equivalence of projections. Recall also that the inner groupoid $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is totally intransitive. All morphisms between the objects of the diagram are smooth maps with respect to their Banach manifold structures. So, one can consider (3.6) as a short exact sequence of Banach-Lie groupoids.

Since the construction of the algebroid of a Banach-Lie groupoid has functorial property one obtains from (3.6) the short exact sequence of corresponding Banach-Lie algebroids

$$\begin{array}{ccccc} \mathcal{A}\mathcal{J}(\mathfrak{M}) & \xrightarrow{\iota} & \mathcal{A}\mathcal{G}(\mathfrak{M}) & \xrightarrow{a} & T\mathcal{L}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}). \end{array} \quad (3.8)$$

Note here that the tangent bundle $T\mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is the algebroid of the pair groupoid $\mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proposition 3.1. *The short exact sequences (3.5) and (3.8) of the Banach vector bundles are isomorphic in a canonical way.*

Proof. One has a canonical inclusion of the bundles

$$\begin{array}{ccc}
\mathcal{G}(\mathfrak{M}) & \xrightarrow{\iota} & \mathcal{M}^L(\mathfrak{M}) \\
\mathbf{s} \downarrow & & \downarrow \pi_{\mathcal{M}} \\
\mathcal{L}(\mathfrak{M}) & \xrightarrow{id} & \mathcal{L}(\mathfrak{M})
\end{array} \tag{3.9}$$

defined by $\iota(x) := (x, \mathbf{s}(x))$, where the source map fibre $\mathbf{s}^{-1}(q)$ of $q \in \mathcal{L}(\mathfrak{M})$ is an open subset of the fibre $\pi_{\mathcal{M}}^{-1}(q) = \mathfrak{M}q$. Thus one obtains the isomorphisms $T_q(\mathbf{s}^{-1}(q)) \cong \mathfrak{M}q$ of the space $T_q(\mathbf{s}^{-1}(q))$ tangent to $\mathbf{s}^{-1}(q)$ at q with the fibre $\pi_{\mathcal{M}}^{-1}(q)$, for details see Proposition 5.2 in [18]. From the above we conclude that $\mathcal{M}^L(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is isomorphic with the algebroid $\mathcal{AG}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

The inner subgroupoid $\mathcal{J}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ can be considered as a bundle $\mathbf{s} : \mathcal{J}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ of groups $\mathbf{s}^{-1}(q) \cap \mathbf{t}^{-1}(q) = G(q\mathfrak{M}q)$. Similarly to (3.9) one has

$$\begin{array}{ccc}
\mathcal{J}(\mathfrak{M}) & \xrightarrow{\iota} & \mathcal{A}(\mathfrak{M}) \\
\mathbf{s} \downarrow & & \downarrow \pi_{\mathcal{A}} \\
\mathcal{L}(\mathfrak{M}) & \xrightarrow{id} & \mathcal{L}(\mathfrak{M})
\end{array} \tag{3.10}$$

where $\mathbf{s}^{-1}(q) = G(q\mathfrak{M}q)$ is an open subset of $q\mathfrak{M}q = \pi_{\mathcal{A}}^{-1}(q)$. So, using the same arguments as for (3.9), we conclude that $\mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is isomorphic with the algebroid $\mathcal{AJ}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of the inner subgroupoid.

Let $]-\varepsilon, \varepsilon[\ni t \mapsto q(t) \in \Pi_q$ be a smooth curve through the point $q = q(0)$. Because of $a_q := \frac{d}{dt}\varphi_q(q(t))|_{t=0} \in (1-q)\mathfrak{M}q$ one obtains the isomorphism $T_q\mathcal{L}(\mathfrak{M}) \cong (1-q)\mathfrak{M}q$ for any $q \in \mathcal{L}(\mathfrak{M})$. Taking into account that the atlas $(\Pi_q, \varphi_q : \Pi \rightarrow (1-q)\mathfrak{M}q)$, $q \in \mathcal{L}(\mathfrak{M})$, is defined in a canonical way one has the canonical isomorphism between $\mathcal{T}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and $T\mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. \square

Let us also mention that two of the Banach vector bundles included in (3.5) are equipped with some additional structures:

- (i) the bundle $\pi_{\mathcal{M}} : \mathcal{M}^L(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is a bundle of the left \mathfrak{M} -modules;
- (ii) the bundle $\pi_{\mathcal{A}} : \mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is a bundle of W^* -algebras.

All above statements are valid in the case if one takes the subgroupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ instead of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proposition 3.2. *The Atiyah sequence (2.13) is isomorphic with the short exact sequence of the Banach-Lie algebroids*

$$\begin{array}{ccccc}
\mathcal{AJ}_{p_0}(\mathfrak{M}) & \longrightarrow & \mathcal{AG}_{p_0}(\mathfrak{M}) & \longrightarrow & \mathcal{TL}_{p_0}(\mathfrak{M}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M}),
\end{array} \tag{3.11}$$

Proof. Let us denote the G_0 -orbits of $(x, \eta) \in p_0\mathfrak{M}p_0 \times P_0$ and $(\vartheta, \eta) \in \mathfrak{M}p_0 \times P_0$ by $\langle x, \eta \rangle \in p_0\mathfrak{M}p_0 \times_{Ad_{G_0}} P_0$ and $\langle \vartheta, \eta \rangle \in TP_0/G_0$, respectively, and the TG_0 -orbit of $(\vartheta, \eta) \in TP_0$ by $\langle\langle \vartheta, \eta \rangle\rangle \in$

TP_0/TG_0 . The maps

$$I_A : p_0 \mathfrak{M}_{p_0} \times_{Ad_{G_0}} P_0 \ni \langle x, \eta \rangle \mapsto (\eta x \eta^{-1}, \eta \eta^{-1}) \in \mathcal{A}_{p_0}(\mathfrak{M}) \cong \mathcal{AJ}_{p_0}(\mathfrak{M}) \quad (3.12)$$

$$I_{\mathcal{M}} : \frac{\mathfrak{M}_{p_0} \times P_0}{G_0} \ni \langle \vartheta, \eta \rangle \mapsto (\vartheta \eta^{-1}, \eta \eta^{-1}) \in \mathcal{M}_{p_0}^L(\mathfrak{M}) \cong \mathcal{AG}_{p_0}(\mathfrak{M}) \quad (3.13)$$

$$I_{\mathcal{T}} : \frac{\mathfrak{M}_{p_0} \times P_0}{TG_0} \ni \langle \langle \vartheta, \eta \rangle \rangle \mapsto ((1 - \eta \eta^{-1}) \vartheta \eta^{-1}, \eta \eta^{-1}) \in \mathcal{T}_{p_0}(\mathfrak{M}) \cong T\mathcal{L}_{p_0}(\mathfrak{M}) \quad (3.14)$$

define isomorphisms between the corresponding Banach vector bundles appearing in the diagrams (2.13) and (3.11) and they commute with the horizontal arrows of these diagrams. We recall that $\mathcal{L}_{p_0}(\mathfrak{M}) \cong P_0/G_0$. \square

Taking into account Proposition 3.1 and Proposition 3.2 we will call (3.8) (as well as (3.5)) the **Atiyah sequence** of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partially invertible elements of W^* -algebra \mathfrak{M} .

Now following of [18] we will present the formula for Lie bracket $[\mathfrak{X}_1, \mathfrak{X}_2]$ of sections $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma^\infty(\mathcal{AG}(\mathfrak{M})) \cong \Gamma^\infty(\mathcal{M}^L(\mathfrak{M}))$ of the bundle $\mathcal{M}^L(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$.

To this end let us recall that the one-parameter group $L_t \circ L_s = L_{t+s}$ of the left translations $L_t : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ by the definition has the following properties

$$\begin{aligned} (i) \quad & L_t(xy) = L_t(x)y \\ (ii) \quad & \mathbf{s} \circ L_t = \mathbf{s} \\ (iii) \quad & \mathbf{t} \circ L_t = \lambda_t \circ \mathbf{t}, \end{aligned} \quad (3.15)$$

where $\mathbf{s}(x) = \mathbf{t}(y)$ and the one-parameter group $\lambda_t : \mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ is defined in a unique way by L_t .

From (3.15) and from $\mathbf{t} \circ \sigma_p = id_{\Pi_p}$ we obtain

$$\mathbf{t}(L_t(\sigma_p(q))) = (\lambda_t \circ \mathbf{t} \circ \sigma_p)(q) = \lambda_t(q) = \mathbf{t}(\sigma_p(\lambda_t(q))), \quad (3.16)$$

$$\mathbf{s}(L_t(\sigma_p(q))) = \mathbf{s}(\sigma_p(\lambda_t(q))) = p. \quad (3.17)$$

It follows from (3.16) and (3.17) that $L_t(\sigma_p(q))$ and $\sigma_p(\lambda_t(q))$ belong to $\mathbf{s}^{-1}(p) \cap \mathbf{t}^{-1}(\lambda_t(q))$. Thus there exists uniquely defined $c_p(q, t) \in G(p\mathfrak{M}p)$ such that

$$L_t(\sigma_p(q)) = \sigma_p(\lambda_t(q))c_p(t, q). \quad (3.18)$$

From $L_t \circ L_s = L_{t+s}$ it follows that the cocycle property

$$c_p(q, t+s) = c_p(\lambda_t(q), s)c_p(q, t) \quad (3.19)$$

is valid for $c_p :]-\varepsilon, \varepsilon[\times \Pi_p \rightarrow G(p\mathfrak{M}p)$.

Proposition 3.3. *One has the following equalities:*

$$c_p(q, t) = p L_t(\sigma_p(q)) \quad (3.20)$$

$$c_p(q, t) = z_{pp_0}(t)z_{pp_0}^{-1} \quad (3.21)$$

$$z_{p\hat{p}}(t) = z_{pp_0}(t)\tilde{z}_{\hat{p}p_0}^{-1} \quad (3.22)$$

$$b_p = \frac{d}{dt}c_p(q, t)|_{t=0} \quad (3.23)$$

$$b_{p\tilde{p}} = b_p z_{p\tilde{p}} \quad (3.24)$$

where $z_{pp_0}(t)$ and $z_{p\tilde{p}}(t)$ are defined by

$$L_t(\eta) = (p + y_p(t))z_{pp_0}(t) \quad (3.25)$$

and by

$$L_t(x) = (p + y_p(t))z_{p\tilde{p}}(t)(\tilde{p} + \tilde{y}_{\tilde{p}}(t))^{-1}. \quad (3.26)$$

For the definition of b_p and $b_{p\tilde{p}}$ see (2.24) and (2.38), respectively.

Proof. Multiplying of (3.18) on the left hand side by p and using the equality $p\sigma_p(q) = p$ we obtain (3.20). From (3.15) we have

$$L_t(\eta) = L_t(\sigma_p(q))z_{pp_0}. \quad (3.27)$$

Comparing (3.25) with (3.27) and using (3.18) we obtain

$$L_t(\eta) = \sigma_p(\lambda_t(q))z_{pp_0}(t) = L_t(\sigma_p(q))z_{pp_0} = \sigma_p(\lambda_t(q))c_p(t, q)z_{pp_0}. \quad (3.28)$$

Canceling $\sigma_p(\lambda_t(q))$ in (3.28) we obtain (3.21). The equality (3.22) follows from $L_t(x) = L_t(\eta)\xi^{-1}$ and from (3.25) and (3.26). Equalities (3.23) and (3.24) are proved by the straightforward checking. \square

From (3.15) it follows that the vector field $\tilde{\mathfrak{X}} \in \Gamma^\infty(T\mathcal{G}(\mathfrak{M}))$ tangent to the flow $L_t : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ satisfies

$$\tilde{\mathfrak{X}}(xy) = TR_y(x)\tilde{\mathfrak{X}}(x). \quad (3.29)$$

Hence one has the one-to-one correspondence

$$\tilde{\mathfrak{X}}(x) = TR_x(q)\mathfrak{X}(q), \quad (3.30)$$

where $q = \mathbf{t}(x) = xx^{-1}$, between $\tilde{\mathfrak{X}}$ and its restriction \mathfrak{X} to $\mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ which, because of (3.9), is a section $\mathfrak{X} \in \Gamma^\infty(\mathcal{M}^L(\mathfrak{M}))$ of the bundle $\mathcal{M}^L(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. As we see from the definition

$$[\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2](x) := \lim_{t \rightarrow 0} \left(\tilde{\mathfrak{X}}_2(x) - TL_t^1(L_{-t}^1(x))\tilde{\mathfrak{X}}_2(L_t^1(x)) \right) \quad (3.31)$$

the Lie bracket $[\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2]$ of vector fields $\tilde{\mathfrak{X}}_1$ and $\tilde{\mathfrak{X}}_2$ tangent to L_t^1 and L_t^2 , respectively, satisfies the conditions (3.15), and thus the property (3.29). So, one can define the Lie bracket $[\mathfrak{X}_1, \mathfrak{X}_2]$ of $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma^\infty(\mathcal{M}^L(\mathfrak{M}))$ restricting $[\tilde{\mathfrak{X}}_1, \tilde{\mathfrak{X}}_2]$ to $\mathcal{L}(\mathfrak{M})$.

Let us now express the Lie bracket (3.31) in the coordinates $(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$. Using (3.26) we have

$$\begin{aligned} \tilde{\mathfrak{X}}(f) &= \frac{d}{dt}(f \circ L_t)|_{t=0} = \left\langle \frac{\partial f}{\partial y_p}, \frac{dy_p(t)}{dt} \Big|_{t=0} \right\rangle + \left\langle \frac{\partial f}{\partial z_{p\tilde{p}}}, \frac{dz_{p\tilde{p}}(t)}{dt} \Big|_{t=0} \right\rangle + \left\langle \frac{\partial f}{\partial \tilde{y}_{\tilde{p}}}, \frac{d\tilde{y}_{\tilde{p}}(t)}{dt} \Big|_{t=0} \right\rangle = \\ &= \left\langle \frac{\partial f}{\partial y_p}, a_p \right\rangle + \left\langle \frac{\partial f}{\partial z_{p\tilde{p}}}, b_p z_{p\tilde{p}} \right\rangle \end{aligned} \quad (3.32)$$

for $f \in C^\infty(\mathcal{G}(\mathfrak{M}))$, where in order to obtain the last equality in (3.32) we have used (2.23), (2.24) and (3.24) and taken into account the independence of $\tilde{y}_{\tilde{p}}(t) = \text{const}$ on $t \in]-\varepsilon, \varepsilon[$. Hence the vector field $\tilde{\mathfrak{X}}$ tangent to the left translation flow L_t written in the coordinates $(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}})$ assumes the following form

$$\tilde{\mathfrak{X}} = a_p \frac{\partial}{\partial y_p} + b_p z_{p\tilde{p}} \frac{\partial}{\partial z_{p\tilde{p}}}, \quad (3.33)$$

where $a_p(y_p) \in (1-p)\mathfrak{M}p$ and $b_p(y_p) \in p\mathfrak{M}p$. Restricting $\tilde{\mathfrak{X}}$ to P_0 and to $\mathcal{L}(\mathfrak{M})$ we obtain

$$\mathcal{V} = a_p \frac{\partial}{\partial y_p} + b_p z_{pp_0} \frac{\partial}{\partial z_{pp_0}} \quad (3.34)$$

and

$$\mathfrak{X} = a_p \frac{\partial}{\partial y_p} + b_p \frac{\partial}{\partial z_{pp_0}}, \quad (3.35)$$

respectively, where $\mathcal{V} \in \Gamma_{G_0}^\infty TP_0$ is G_0 -invariant vector field on P_0 and $\mathfrak{X} \in \Gamma^\infty \mathcal{M}^L(\mathfrak{M})$ is a section of $\mathcal{M}^L(\mathfrak{M})$ defined by (3.30). Sections of the vector bundles $T\mathcal{G}(\mathfrak{M})$, TP_0 and $\mathcal{M}^L(\mathfrak{M})$ presented above give the equivalent coordinate representations of sections of the algebroid $\mathcal{AG}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. Note here that the transition map between (a_p, b_p) and $(a_{p'}, b_{p'})$ is given in (2.32) and (2.33).

Proposition 3.4. (i) *The anchor map $a : \mathcal{AL}(\mathfrak{M}) \rightarrow T\mathcal{L}(\mathfrak{M})$ acts on (3.35) as follows:*

$$a(\mathfrak{X}) = a_p \frac{\partial}{\partial y_p}; \quad (3.36)$$

(ii) *The vertical part of (3.35) is given by $b_p \frac{\partial}{\partial z_{pp_0}}$;*

(iii) *The Lie bracket of $\mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma^\infty \mathcal{M}^L(\mathfrak{M})$ assumes the form*

$$[\mathfrak{X}_1, \mathfrak{X}_2] = a_p \frac{\partial}{\partial y_p} + b_p \frac{\partial}{\partial z_{pp}}, \quad (3.37)$$

where

$$a_p = \left\langle \frac{\partial a_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial a_{1p}}{\partial y_p}, a_{2p} \right\rangle \quad (3.38)$$

and

$$b_p = \left\langle \frac{\partial b_{2p}}{\partial y_p}, a_{1p} \right\rangle - \left\langle \frac{\partial b_{1p}}{\partial y_p}, a_{2p} \right\rangle + [b_{2p}, b_{1p}]. \quad (3.39)$$

Proof. The proof can be done by the straightforward verification. \square

Below we will also write $\mathcal{V} \in \Gamma_{G_0}^\infty TP_0$ using the global coordinate $\eta \in P_0$:

$$\mathcal{V} = v \frac{\partial}{\partial \eta}, \quad (3.40)$$

where $v : P_0 \rightarrow \mathfrak{M}p_0$ satisfies $v(\eta g) = v(\eta)g$ for $g \in G_0$. In this representation the Lie bracket of $\mathcal{V}_1, \mathcal{V}_2 \in \Gamma_{G_0}^\infty TP_0$ is given by

$$[\mathcal{V}_1, \mathcal{V}_2] = \left(\left\langle \frac{\partial v_2}{\partial \eta}, v_1 \right\rangle - \left\langle \frac{\partial v_1}{\partial \eta}, v_2 \right\rangle \right) \frac{\partial}{\partial \eta}. \quad (3.41)$$

Note here that $\frac{\partial v}{\partial \eta}(\eta) : \mathfrak{M}p_0 \rightarrow \mathfrak{M}p_0$.

4 Predual Atiyah sequence of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

We recall that, by definition, W^* -algebra \mathfrak{M} is a C^* -algebra possessing a predual Banach space \mathfrak{M}_* (i.e. $(\mathfrak{M}_*)^* = \mathfrak{M}$) which is defined in the unique way by the structure of \mathfrak{M} , e.g. see [20]. Hence, to the Banach vector bundles (algebroids) $\mathcal{A}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$, $\mathcal{AG}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and $T\mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ appearing in (3.8) canonically correspond their predual counterparts $\mathcal{A}_*(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$, $\mathcal{A}_*\mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and $T_*\mathcal{L}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$. These bundles are Banach quasi subbundles

$$\mathcal{A}_*(\mathfrak{M}) \subset \mathcal{A}^*(\mathfrak{M}), \quad \mathcal{A}_*\mathcal{G}(\mathfrak{M}) \subset \mathcal{A}^*\mathcal{G}(\mathfrak{M}) \quad \text{and} \quad T_*\mathcal{L}(\mathfrak{M}) \subset T^*\mathcal{L}(\mathfrak{M}) \quad (4.1)$$

of the corresponding dual bundles, i.e. their fibres are Banach subspaces but without Banach complements.

The bundle morphisms a^* and ι^* dual to the ones from (3.5) preserve the predual subbundles (4.1). Thus their restrictions a_* and ι_* define the short exact sequence

$$\begin{array}{ccccc} T_*\mathcal{L}(\mathfrak{M}) & \xrightarrow{a_*} & \mathcal{A}_*\mathcal{G}(\mathfrak{M}) & \xrightarrow{\iota_*} & \mathcal{A}_*\mathcal{J}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}(\mathfrak{M}) \end{array} \quad (4.2)$$

of Banach bundles which will be the main object of our considerations in this section. Dualizing (4.2) we return to (3.8). So, we will call (4.2) the **predual Atiyah sequence** of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

In order to make the above statements precise we note that the left action $L_a x := ax$ (the right action $R_a x := xa$) of \mathfrak{M} on itself generates the right action (left action) of \mathfrak{M} on the dual \mathfrak{M}^*

$$\begin{aligned} \langle R_a^* \varphi, x \rangle &:= \langle \varphi, ax \rangle \\ \langle L_a^* \varphi, x \rangle &:= \langle \varphi, xa \rangle \end{aligned} \quad (4.3)$$

where $a, x \in \mathfrak{M}$ and $\varphi \in \mathfrak{M}^*$. In a sequel we will write φa and $a\varphi$ instead of $R_a^* \varphi$ and $L_a^* \varphi$. Using this notation we mention the following isomorphisms

$$\begin{aligned} (\mathfrak{M}p)^* &\cong p\mathfrak{M}^*, \\ (q\mathfrak{M})^* &\cong \mathfrak{M}^*q, \\ (q\mathfrak{M}p)^* &\cong p\mathfrak{M}^*q, \end{aligned} \quad (4.4)$$

where $p, q \in \mathcal{L}(\mathfrak{M})$ and $(\mathfrak{M}p)^*$, $(q\mathfrak{M})^*$ and $(q\mathfrak{M}p)^*$ are duals of corresponding Banach subspaces of \mathfrak{M} . Since the predual Banach space \mathfrak{M}_* is a Banach subspace $\mathfrak{M}_* \subset \mathfrak{M}^*$ of \mathfrak{M}^* invariant with respect to R_a^* and L_a^* one has the respective predual maps $R_{*a} : \mathfrak{M}_* \rightarrow \mathfrak{M}_*$ and $L_{*a} : \mathfrak{M}_* \rightarrow \mathfrak{M}_*$ defined as restrictions of R_a^* and L_a^* to \mathfrak{M}_* . Hence, similarly to (4.4) one has the isomorphisms

$$\begin{aligned} (\mathfrak{M}p)_* &\cong p\mathfrak{M}_*, \\ (q\mathfrak{M})_* &\cong \mathfrak{M}_*q, \\ (q\mathfrak{M}p)_* &\cong p\mathfrak{M}_*q. \end{aligned} \quad (4.5)$$

The actions of G_0 on $T_*P_0 \cong p_0\mathfrak{M}_* \times P_0$ and on $p_0\mathfrak{M}_*p_0 \times P_0$, predual to the action of G_0 on $TP_0 \cong \mathfrak{M}p_0 \times P_0$ and on $p_0\mathfrak{M}p_0 \times P_0$ are defined as follows

$$p_0\mathfrak{M}_* \times P_0 \ni (\varphi, \eta) \mapsto \Sigma_{*g}(\varphi, \eta) := (g^{-1}\varphi, \eta g), \quad (4.6)$$

$$p_0\mathfrak{M}_*p_0 \times P_0 \ni (\mathcal{X}, \eta) \mapsto (g^{-1}\mathcal{X}g, \eta g). \quad (4.7)$$

Let us define a group structure on the precotangent bundle T_*G_0 identifying it with the semidirect product $p_0\mathfrak{M}_*p_0 \rtimes_{Ad_{G_0}^*} G_0$ of groups G_0 and $p_0\mathfrak{M}_*p_0$, i.e. the group product on T_*G_0 is given by

$$(\mathcal{X}, g) \star (\mathcal{Y}, h) := (\mathcal{X} + g\mathcal{Y}g^{-1}, gh). \quad (4.8)$$

The precotangent group T_*G_0 acts on $T_*P_0 \cong \mathfrak{M}_*p_0 \times P_0$ in the following way

$$(\varphi, \eta) \mapsto (g^{-1}(\varphi + \mathcal{X}\eta^{-1}), \eta g). \quad (4.9)$$

The next proposition is the predual version of Proposition 3.2.

Proposition 4.1. *The predual Atiyah sequence of the principal bundle $P_0 \rightarrow P_0/G_0$*

$$\begin{array}{ccccc} T_*(P_0/G_0) & \xrightarrow{a_*} & T_*P_0/G_0 & \xrightarrow{\iota_*} & p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0 \\ \downarrow & & \downarrow & & \downarrow \\ P_0/G_0 & \xrightarrow{\sim} & P_0/G_0 & \xrightarrow{\sim} & P_0/G_0, \end{array} \quad (4.10)$$

is isomorphic with the short exact sequence

$$\begin{array}{ccccc} \mathcal{T}_*\mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{a_*} & \mathcal{A}_*\mathcal{G}_{p_0}(\mathfrak{M}) & \xrightarrow{\iota_*} & \mathcal{A}_*\mathcal{J}_{p_0}(\mathfrak{M}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M}) & \xrightarrow{\sim} & \mathcal{L}_{p_0}(\mathfrak{M}), \end{array} \quad (4.11)$$

i.e. the predual Atiyah sequence of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$.

Proof. The isomorphism between sequences (4.10) and (4.11) is given by the following isomorphisms

$$I_{*\mathcal{T}} : \frac{\mathfrak{M}_*p_0 \times P_0}{T_*G_0} \ni \langle \langle \varphi, \eta \rangle \rangle \mapsto (\eta\varphi(1 - \eta\eta^{-1}), \eta\eta^{-1}) \in \mathcal{T}_{*p_0}(\mathfrak{M}) \cong T_*\mathcal{L}_{p_0}(\mathfrak{M}) \quad (4.12)$$

$$I_{*\mathcal{M}} : \frac{\mathfrak{M}_*p_0 \times P_0}{G_0} \ni \langle \varphi, \eta \rangle \mapsto (\eta\varphi, \eta\eta^{-1}) \in \mathcal{M}_{*p_0}^L(\mathfrak{M}) \cong \mathcal{A}_*\mathcal{G}_{p_0}(\mathfrak{M}) \quad (4.13)$$

$$I_{*\mathcal{A}} : p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0 \ni \langle \mathcal{X}, \eta \rangle \mapsto (\eta\mathcal{X}\eta^{-1}, \eta\eta^{-1}) \in \mathcal{A}_{*p_0}(\mathfrak{M}) \cong \mathcal{A}_*\mathcal{J}_{p_0}(\mathfrak{M}) \quad (4.14)$$

of the corresponding Banach vector bundles, where by $\langle \langle \varphi, \eta \rangle \rangle$, $\langle \varphi, \eta \rangle$, $\langle \mathcal{X}, \eta \rangle$ we denote equivalence classes defined by the corresponding group actions. We recall that $\mathcal{L}_{p_0}(\mathfrak{M}) \cong P_0/G_0$. \square

The action tangent to the action (4.6), after taking into account the bundle isomorphism $T(T_*P_0) \cong p_0\mathfrak{M}_* \times \mathfrak{M}p_0 \times (p_0\mathfrak{M}_* \times P_0)$, is the following

$$T\Sigma_{*g}(\varphi, \eta)\xi_{(\varphi, \eta)} = (g^{-1}\theta, vg, g^{-1}\varphi, \eta g), \quad (4.15)$$

where $\xi_{(\varphi, \eta)} = (\theta, v, \varphi, \eta) \in T_{(\varphi, \eta)}(p_0\mathfrak{M}_* \times P_0) \cong (p_0\mathfrak{M}_* \times \mathfrak{M}p_0) \times \{(\varphi, \eta)\}$.

Let $\pi_* := pr_2 : T_*P_0 \cong p_0\mathfrak{M}_* \times P_0 \rightarrow P_0$ be the projection on the bundle base. Since $T\pi_* \circ T\Sigma_{*g} = \Sigma_g \circ T\pi_*$ one easily sees that the canonical 1-form

$$\langle \gamma_{(\varphi, \eta)}, \xi_{(\varphi, \eta)} \rangle := \langle (\varphi, \eta), T\pi^*(\varphi, \eta)\xi_{(\varphi, \eta)} \rangle = \langle \varphi, v \rangle \quad (4.16)$$

and, thus the 2-form $\omega := d\gamma$ are invariant with respect to (4.15). For $\xi_{(\varphi, \eta)}^1, \xi_{(\varphi, \eta)}^2 \in T_{(\varphi, \eta)}(p_0\mathfrak{M}_* \times P_0)$ one has

$$\omega_{(\varphi, \eta)}(\xi_{(\varphi, \eta)}^1, \xi_{(\varphi, \eta)}^2) = \langle \theta_1, v_2 \rangle - \langle \theta_2, v_1 \rangle. \quad (4.17)$$

The image of the bundle monomorphism

$$\omega_{(\varphi, \eta)}(\xi_{(\varphi, \eta)}, \cdot) : T_{(\varphi, \eta)}(p_0\mathfrak{M}_* \times P_0) \rightarrow T_{(\varphi, \eta)}^*(p_0\mathfrak{M}_* \times P_0) \quad (4.18)$$

is a Banach subspace $\mathfrak{M}p_0 \times p_0\mathfrak{M}_* \times \{(\varphi, \eta)\} \subsetneq \mathfrak{M}p_0 \times (\mathfrak{M}p_0)^* \times \{(\varphi, \eta)\} \cong T_{(\varphi, \eta)}^*(p_0\mathfrak{M}_* \times P_0)$ of the cotangent Banach space at (φ, η) . Hence, the 2-form ω is only a weak symplectic form on $T_*(P_0)$ in sense of the definition presented for example in [16]. So, the fibre monomorphisms (4.18) define the bundle quasi immersion $\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$, i.e. $T^\flat(T_*P_0) := \flat(T(T_*P_0))$ is a subbundle of the cotangent bundle $T^*(T_*P_0)$ but without the split rang in general, i.e. it is a quasi Banach subbundle.

Now, for $x \in p_0\mathfrak{M}p_0$ we define $\xi^x \in \Gamma^\infty T(T_*P_0)$ by

$$\xi^x(f)(\varphi, \eta) := \frac{d}{dt} f(\Sigma_{*exp(tx)}(\varphi, \eta))|_{t=0} \quad (4.19)$$

where $f \in C^\infty(T_*P_0)$. One has

$$\omega(\xi^x, \cdot) = -d\langle \gamma, \xi^x \rangle = -d\langle J_0, x \rangle. \quad (4.20)$$

The last term in (4.20) contains the momentum map $J_0 : T_*P_0 \rightarrow p_0\mathfrak{M}_*p_0$ defined as follows

$$J_0(\varphi, \eta) := \varphi\eta, \quad (4.21)$$

i.e. by definition, for any $x \in p_0\mathfrak{M}p_0$, one has $\langle J_0(\varphi, \eta), x \rangle := \langle \varphi, \eta x \rangle$. We note that the equivariance property

$$J_0 \circ \Sigma_{*g} = Ad_{g^{-1}}^* \circ J_0 \quad (4.22)$$

with respect to the group G_0 is valid for (4.21).

For any $f \in C^\infty(T_*P_0)$, one has $\frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)^*$ and $\frac{\partial f}{\partial \varphi}(\varphi, \eta) \in (p_0\mathfrak{M}_*)^* \cong \mathfrak{M}p_0$. Therefore, we can define the bracket

$$\{f, g\} := \left\langle \frac{\partial g}{\partial \eta}, \frac{\partial f}{\partial \varphi} \right\rangle - \left\langle \frac{\partial f}{\partial \eta}, \frac{\partial g}{\partial \varphi} \right\rangle \quad (4.23)$$

of $f, g \in C^\infty(T_*P_0)$, which is bilinear, anti-symmetric and satisfies the Leibniz property. However, for arbitrary smooth functions on T_*P_0 the Jacobi identity for (4.23) is not fulfilled. For this reason we define the function space

$$\mathcal{P}^\infty(T_*P_0) := \left\{ f \in C^\infty(T_*P_0) : \frac{\partial f}{\partial \eta}(\varphi, \eta) \in (\mathfrak{M}p_0)_* \subset (\mathfrak{M}p_0)^* \right\}. \quad (4.24)$$

Proposition 4.2. *The function space $(\mathcal{P}^\infty(T_*P_0), \{\cdot, \cdot\})$ is a Poisson algebra with respect to the bracket (4.23). The derivation $\{f, \cdot\}$ defined by $f \in \mathcal{P}^\infty(T_*P_0)$ is a vector field $\xi_f \in \Gamma^\infty T(T_*P_0)$ satisfying*

$$\omega(\xi_f, \cdot) = -df, \quad (4.25)$$

i.e. it is a Hamiltonian with respect to the weak symplectic form (4.17).

Proof. At first we observe that $\mathcal{P}^\infty(T_*P_0)$ is a subalgebra of the associative algebra of all smooth functions $C^\infty(T_*P_0)$. In order to prove that $\mathcal{P}^\infty(T_*P_0)$ is closed with respect to the bracket (4.16) we note that

$$\left\langle \frac{\partial^2 f}{\partial \eta \partial \varphi}; \dot{\eta}, \dot{\varphi} \right\rangle = \left\langle \frac{\partial^2 f}{\partial \varphi \partial \eta}; \dot{\varphi}, \dot{\eta} \right\rangle \quad (4.26)$$

and

$$\left\langle \frac{\partial^2 f}{\partial \eta^2}; \dot{\eta}_1, \dot{\eta}_2 \right\rangle = \left\langle \frac{\partial^2 f}{\partial \eta^2}; \dot{\eta}_2, \dot{\eta}_1 \right\rangle \quad (4.27)$$

for any $\dot{\eta}, \dot{\eta}_1, \dot{\eta}_2 \in \mathfrak{M}_{p_0}$ and $\dot{\varphi} \in p_0\mathfrak{M}_*$. From (4.23) we find that for any $\dot{\eta} \in \mathfrak{M}_{p_0}$ one has

$$\begin{aligned} \left\langle \frac{\partial \{f, g\}}{\partial \eta}, \dot{\eta} \right\rangle &= \left\langle \frac{\partial^2 g}{\partial \eta^2}; \frac{\partial f}{\partial \varphi}, \dot{\eta} \right\rangle - \left\langle \frac{\partial^2 f}{\partial \eta^2}; \frac{\partial g}{\partial \varphi}, \dot{\eta} \right\rangle + \left\langle \frac{\partial^2 f}{\partial \eta \partial \varphi}; \dot{\eta}, \frac{\partial g}{\partial \eta} \right\rangle - \left\langle \frac{\partial^2 g}{\partial \eta \partial \varphi}; \dot{\eta}, \frac{\partial f}{\partial \eta} \right\rangle = \\ &= \left\langle \frac{\partial^2 g}{\partial \eta^2}; \frac{\partial f}{\partial \varphi}, \dot{\eta} \right\rangle - \left\langle \frac{\partial^2 f}{\partial \eta^2}; \frac{\partial g}{\partial \varphi}, \dot{\eta} \right\rangle + \left\langle \frac{\partial^2 f}{\partial \varphi \partial \eta}; \frac{\partial g}{\partial \eta}, \dot{\eta} \right\rangle - \left\langle \frac{\partial^2 g}{\partial \varphi \partial \eta}; \frac{\partial f}{\partial \eta}, \dot{\eta} \right\rangle. \end{aligned} \quad (4.28)$$

Since $\frac{\partial^2 f}{\partial \eta^2}(\varphi, \eta) \in \mathcal{L}(\mathfrak{M}_{p_0}, p_0\mathfrak{M}_*)$, $\frac{\partial^2 f}{\partial \varphi \partial \eta}(\varphi, \eta) \in \mathcal{L}(p_0\mathfrak{M}_*, \mathfrak{M}_{p_0})$, $\frac{\partial f}{\partial \varphi}(\varphi, \eta), \frac{\partial g}{\partial \varphi}(\varphi, \eta) \in (p_0\mathfrak{M}_*)^* = \mathfrak{M}_{p_0}$ and $\frac{\partial f}{\partial \eta}(\varphi, \eta), \frac{\partial g}{\partial \eta}(\varphi, \eta) \in p_0\mathfrak{M}_*$ we find that $\left\langle \frac{\partial^2 g}{\partial \eta^2}; \frac{\partial f}{\partial \varphi}, \cdot \right\rangle - \left\langle \frac{\partial^2 f}{\partial \eta^2}; \frac{\partial g}{\partial \varphi}, \cdot \right\rangle + \left\langle \frac{\partial^2 f}{\partial \varphi \partial \eta}; \frac{\partial g}{\partial \eta}, \cdot \right\rangle - \left\langle \frac{\partial^2 g}{\partial \varphi \partial \eta}; \frac{\partial f}{\partial \eta}, \cdot \right\rangle \in p_0\mathfrak{M}_*$. Thus and from (4.28) we see that $\frac{\partial \{f, g\}}{\partial \eta}(\varphi, \eta) \in p_0\mathfrak{M}_*$. So, we have proved that $\{f, g\} \in \mathcal{P}^\infty(T_*P_0)$.

For proving the Jacobi identity for the bracket (4.23) we take

$$\begin{aligned} \{\{f, g\}, h\} &= \left\langle \frac{\partial h}{\partial \eta}, \frac{\partial \{f, g\}}{\partial \varphi} \right\rangle - \left\langle \frac{\partial \{f, g\}}{\partial \eta}, \frac{\partial h}{\partial \varphi} \right\rangle = \\ &= \left\langle \frac{\partial^2 g}{\partial \varphi \partial \eta}; \frac{\partial h}{\partial \eta}, \frac{\partial f}{\partial \varphi} \right\rangle + \left\langle \frac{\partial^2 f}{\partial \varphi^2}; \frac{\partial h}{\partial \eta}, \frac{\partial g}{\partial \eta} \right\rangle - \left\langle \frac{\partial^2 f}{\partial \varphi \partial \eta}; \frac{\partial h}{\partial \eta}, \frac{\partial g}{\partial \varphi} \right\rangle - \left\langle \frac{\partial^2 g}{\partial \varphi^2}; \frac{\partial h}{\partial \eta}, \frac{\partial f}{\partial \eta} \right\rangle + \\ &+ \left\langle \frac{\partial^2 g}{\partial \eta^2}; \frac{\partial h}{\partial \varphi}, \frac{\partial f}{\partial \varphi} \right\rangle + \left\langle \frac{\partial^2 f}{\partial \eta \partial \varphi}; \frac{\partial h}{\partial \varphi}, \frac{\partial g}{\partial \eta} \right\rangle - \left\langle \frac{\partial^2 f}{\partial \eta^2}; \frac{\partial h}{\partial \varphi}, \frac{\partial g}{\partial \varphi} \right\rangle - \left\langle \frac{\partial^2 g}{\partial \eta \partial \varphi}; \frac{\partial h}{\partial \varphi}, \frac{\partial f}{\partial \eta} \right\rangle. \end{aligned} \quad (4.29)$$

Adding cyclic permutations of (4.29) and taking into account (4.26) and (4.27) we find that Jacobi identity for $f, g, h \in \mathcal{P}^\infty(T_*P_0)$ is satisfied. \square

Remark 4.3. (i) The bracket (4.23) after restriction to $\mathcal{P}^\infty(T_*P_0)$ is the Poisson bracket defined by the weak symplectic form (4.17).

(ii) If $f \in \mathcal{P}^\infty(T_*P_0)$ then the equality (4.25) defines a vector field $\xi_f \in \Gamma^\infty T(T^*P_0)$. But if $f \notin \mathcal{P}^\infty(T_*P_0)$ then $\{f, \cdot\}$ is only a section of the bundle $T^{**}(T_*P_0)$ which contains $T(T_*P_0)$ as a quasi Banach subbundle, i.e. the bundle inclusion $T(T_*P_0) \hookrightarrow T^{**}(T_*P_0)$ has closed range but without the Banach split.

(iii) $f \in \mathcal{P}^\infty(T_*P_0)$ if and only if $df \in \Gamma^\infty T^\flat(T_*P_0)$, i.e. the Banach subbundle $T^\flat(T_*P_0)$ is defined by $\mathcal{P}^\infty(T_*P_0)$.

Since in a general case $T^\flat(T_*P_0) \subsetneq T^*(T_*P_0)$ the Banach bundle morphism $\# : T^\flat(T_*P_0) \rightarrow T(T_*P_0)$ inverse to $\flat : T(T_*P_0) \hookrightarrow T^*(T_*P_0)$ is not defined on the whole of $T^*(T_*P_0)$. So, following of [7], it will be called a **sub Poisson anchor**. Note here that the bracket (4.23) and $\#$ define Banach algebroid structure on $T^\flat(T_*P_0)$.

Using the fibre bundle isomorphisms:

$$\begin{aligned} T(T_*P_0) &\cong p_0\mathfrak{M}_* \times \mathfrak{M}p_0 \times p_0\mathfrak{M}_* \times P_0 \\ T^*(T_*P_0) &\cong \mathfrak{M}p_0 \times (\mathfrak{M}p_0)^* \times p_0\mathfrak{M}_* \times P_0 \\ T^\flat(T_*P_0) &\cong \mathfrak{M}p_0 \times p_0\mathfrak{M}_* \times p_0\mathfrak{M}_* \times P_0 \end{aligned} \quad (4.30)$$

we introduce the following notations:

$$(\dot{\varphi}, \dot{\eta}, \varphi, \eta) \in p_0\mathfrak{M}_* \times \mathfrak{M}p_0 \times p_0\mathfrak{M}_* \times P_0 \quad (4.31)$$

$$(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) \in \mathfrak{M}p_0 \times p_0\mathfrak{M}_* \times p_0\mathfrak{M}_* \times P_0 \quad (4.32)$$

for the coordinates of the elements of $T(T_*P_0)$ and $T^\flat(T_*P_0)$, respectively. The sub Poisson anchor $\#_1 : T^\flat(T_*P_0) \rightarrow T(T_*P_0)$ in the coordinates (5.26) and (4.32) assumes the form

$$\#_1(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) = (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, \varphi, \eta) \quad (4.33)$$

Let us denote by $\mathcal{P}_{G_0}^\infty(T_*P_0) \subset \mathcal{P}^\infty(T_*P_0)$ the subalgebra of G_0 -invariant functions. From G_0 -invariance of the bracket (4.23) it follows that $\mathcal{P}_{G_0}^\infty(T_*P_0)$ is a Poisson subalgebra of $\mathcal{P}^\infty(T_*P_0)$. We will identify $f \in \mathcal{P}_{G_0}^\infty(T_*P_0)$ with a function on the quotient space T_*P_0/G_0 which can be considered as a Banach vector bundle $(\mathfrak{M}p_0)_* \times_{G_0} P_0$ associated with the G_0 -principal bundle $P_0 \rightarrow P_0/G_0$. Thus the quotient projection $Q_0 : T_*P_0 \rightarrow T_*P_0/G_0$ is a submersion, see 6.5.1 in [4]. Therefore one can consider $\mathcal{P}_{G_0}^\infty(T_*P_0)$ as a subalgebra $\mathcal{P}^\infty(T_*P_0/G_0)$ of the algebra $C^\infty(T_*P_0/G_0)$. Hence the Poisson bracket $\{F, G\}_{G_0}$ of $F, G \in \mathcal{P}^\infty(T_*P_0/G_0)$ one defines as follows

$$\{F, G\}_{G_0} := \{F \circ Q_0, G \circ Q_0\}. \quad (4.34)$$

In the case of T_*P_0/G_0 we can define $T^\flat(T_*P_0/G_0)$ as the bundle of germs of 1-forms df , where $f \in \mathcal{P}^\infty(T_*P_0/G_0)$. We note that one has the following isomorphisms:

$$T_{[(\varphi, \eta)]}(T_*P_0/G_0) \cong (\mathfrak{M}p_0)_* \times (1 - p_0)\mathfrak{M}p_0 \times \{[(\varphi, \eta)]\} \quad (4.35)$$

$$T_{[(\varphi, \eta)]}^*(T_*P_0/G_0) \cong \mathfrak{M}p_0 \times p_0\mathfrak{M}^*(1 - p_0) \times \{[(\varphi, \eta)]\} \quad (4.36)$$

$$T_{[(\varphi, \eta)]}^\flat(T_*P_0/G_0) \cong \mathfrak{M}p_0 \times p_0\mathfrak{M}_*(1 - p_0) \times \{[(\varphi, \eta)]\} \quad (4.37)$$

where $[(\varphi, \eta)] \in T_*P_0/G_0$. From (4.36) and (4.37) we see that $T^\flat(T_*P_0/G_0)$ is a proper quasi Banach subbundle of $T^*(T_*P_0/G_0)$.

Similarly as for TP_0 and TP_0/G_0 let us define the atlases on T_*P_0 and T_*P_0/G_0 consistent with their vector bundle structures. We begin from the (φ, η) coordinates on T_*P_0 predual to the coordinates (v, η) on TP_0 . Using (2.25) we find

$$\langle \varphi, v \rangle = \langle \varphi, [a_p + (p + y_p)b_p]_{z_{pp_0}} \rangle = \quad (4.38)$$

$$= \langle z_{pp_0}\varphi(1 - p), a_p \rangle + \langle z_{pp_0}\varphi(p + y_p), b_p \rangle = \langle \alpha_p, a_p \rangle + \langle \beta_p, b_p \rangle,$$

where $\alpha_p \in p\mathfrak{M}_*(1 - p)$ and $\beta_p \in p\mathfrak{M}_*p$ are defined by

$$\alpha_p := z_{pp_0}\varphi(1 - p) \quad (4.39)$$

$$\beta_p := z_{pp_0}\varphi(p + y_p). \quad (4.40)$$

Thus, similarly to (2.27-2.28) we obtain the equalities

$$\alpha_p = (p\eta)\varphi(1 - p) \quad (4.41)$$

$$\beta_p = p\eta\varphi\eta(p\eta)^{-1} \quad (4.42)$$

which together with (1.20) define the chart

$$T_*(\pi_0^{-1}(\Pi_p)) \ni (\varphi, \eta) \mapsto T_*\psi_p(\varphi, \eta) = (\alpha_p, \beta_p, y_p, z_{pp_0}) \quad (4.43)$$

on T_*P_0 , where $p \in \mathcal{L}_{p_0}(\mathfrak{M})$. The dependence inverse to (4.41-4.42) is given by

$$\varphi = z_{pp_0}^{-1}(\alpha_p + \beta_p - \alpha_p y_p) \quad (4.44)$$

$$\eta = (p + y_p)z_{pp_0}. \quad (4.45)$$

The transition map from the coordinates $(\alpha_p, \beta_p, y_p, z_{pp_0})$ to the coordinates $(\alpha_{p'}, \beta_{p'}, y_{p'}, z_{p'p_0})$ is the following

$$\alpha_{p'} = (a + cy_p)(\alpha_p + \beta_p - \alpha_p y_p)(1 - p') \quad (4.46)$$

$$\beta_{p'} = (a + cy_p)\beta_p(a + cy_p)^{-1} \quad (4.47)$$

$$y_{p'} = (b + dy_p)(a + cy_p)^{-1} \quad (4.48)$$

$$z_{p'p_0} = (a + cy_p)z_{pp_0}. \quad (4.49)$$

Quotienting the chart (4.43) by G_0 we obtain on T_*P_0/G_0 the chart

$$T_*(\pi_0^{-1}(\Pi_p))/G_0 \ni \langle \varphi, \eta \rangle \mapsto [T_*\psi_p](\langle \varphi, \eta \rangle) = (\alpha_p, \beta_p, y_p). \quad (4.50)$$

The transition map $[T_*\psi_{p'}] \circ [T_*\psi_p]^{-1} : (\alpha_p, \beta_p, y_p) \mapsto (\alpha_{p'}, \beta_{p'}, y_{p'})$ between these charts is given by (4.46-4.48).

Finally let us note that the momentum map $J_1 : T_*P_0 \rightarrow p_0\mathfrak{M}_*p_0$ in the coordinates $(\alpha_p, \beta_p, y_p, z_{pp_0})$ assumes the form

$$J_1(\alpha_p, \beta_p, y_p, z_{pp_0}) = z_{pp_0}^{-1}\beta_p z_{pp_0} \quad (4.51)$$

Since $p_0\mathfrak{M}_*p_0$ is the predual Banach space of the W^* -algebra $p_0\mathfrak{M}p_0$, see (4.5), the structure of Banach Lie-Poisson space is defined on it in a canonical way. Namely, according to [16], the bracket

$$\{F, G\}_{LP}(\beta) := \left\langle \beta, \left[\frac{\partial F}{\partial \beta}(\beta), \frac{\partial G}{\partial \beta}(\beta) \right] \right\rangle \quad (4.52)$$

is a Lie-Poisson bracket of $F, G \in C^\infty(p_0\mathfrak{M}_*p_0)$. The followig theorem is valid.

Theorem 4.4. (i) *One has the surjective Poisson submersions:*

$$\begin{array}{ccc} & T_*P_0 & \\ Q_0 \swarrow & & \searrow J_1 \\ T_*P_0/G_0 & & p_0\mathfrak{M}_*p_0 \end{array} \quad (4.53)$$

*of the weak symplectic manifold (T_*P_0, ω) on the sub Poisson manifold $(T_*P_0/G_0, \{\cdot, \cdot\}_{G_0})$ and on the Banach Lie-Poisson space $(p_0\mathfrak{M}_*p_0, \{\cdot, \cdot\}_{LP})$.*

(ii) The Poisson subalgebras $J_0^*(C^\infty(p_0\mathfrak{M}_*p_0))$ and $(Q_0)^*(\mathcal{P}^\infty(T_*P_0/G_0)) = \mathcal{P}_{G_0}^\infty(T_*P_0)$ of the Poisson algebra $\mathcal{P}^\infty(T_*P_0)$ are polar one to another with respect to the weak symplectic form ω .

Proof. (i) In order to see that $Q_0 : T_*P_0 \rightarrow T_*P_0/G_0$ is a surjective submersion we note that $T_*P_0/G_0 \rightarrow P_0/G_0$ is a Banach vector bundle associate with the principal bundle $\pi_0 : P_0 \rightarrow P_0/G_0$.

Substituting $\varphi = \beta$ and $\eta = p_0$ into (4.21) we find that $J_1(\beta, p_0) = \beta$. This shows the surjectivity of J_1 .

For $\dot{\varphi} \in p_0\mathfrak{M}_*$ and $\dot{\eta} \in \mathfrak{M}p_0$ one has

$$TJ_1(\varphi, \eta)(\dot{\varphi}, \dot{\eta}) = \frac{\partial J_1}{\partial \varphi}(\varphi, \eta)\dot{\varphi} + \frac{\partial J_1}{\partial \eta}(\varphi, \eta)\dot{\eta} = \dot{\varphi}\eta + \varphi\dot{\eta}. \quad (4.54)$$

Substituting $\eta = p_0$ and $\dot{\eta} = 0$ into (4.54) we obtain that any $x \in p_0\mathfrak{M}_*p_0$ can be written as $x = TJ_1(\varphi, p_0)(\dot{\varphi}, 0) = \dot{\varphi}p_0$. Thus, $TJ_1(\varphi, \eta) : T_{(\varphi, \eta)}(T_*P_0) \rightarrow T_{J_1(\varphi, \eta)}p_0\mathfrak{M}_*p_0$ is a surjection.

Now let us show that $\ker TJ_1(\varphi, \eta)$ has a Banach complement. For this reason we will use the coordinate expression (4.51) for the momentum map J_1 . Differentiating (4.51) we obtain

$$TJ_1(\alpha_p, \beta_p, y_p, z_{pp_0})(\dot{\alpha}_p, \dot{\beta}_p, \dot{y}_p, \dot{z}_{pp_0}) = z_{pp_0}^{-1}(\dot{\beta}_p + \beta_p \dot{z}_{pp_0} z_{pp_0}^{-1} - \dot{z}_{pp_0} z_{pp_0}^{-1} \beta_p) z_{pp_0}, \quad (4.55)$$

where by

$$(\dot{\alpha}_p, \dot{\beta}_p, \dot{y}_p, \dot{z}_{pp_0}) \in p\mathfrak{M}_*(1-p) \oplus p\mathfrak{M}_*p \oplus (1-p)\mathfrak{M}p \oplus p\mathfrak{M}p_0 \quad (4.56)$$

we denote coordinates of the tangent vectors at $(\alpha_p, \beta_p, y_p, z_{pp_0})$. From (4.55) we see that $(\dot{\alpha}_p, \dot{\beta}_p, \dot{y}_p, \dot{z}_{pp_0}) \in \ker TJ_1(\alpha_p, \beta_p, y_p, z_{pp_0})$ iff

$$\dot{\beta}_p = \dot{z}_{pp_0} z_{pp_0}^{-1} \beta_p - \beta_p \dot{z}_{pp_0} z_{pp_0}^{-1}. \quad (4.57)$$

It follows from (4.56) and (4.57) that $\ker TJ_1(\alpha_p, \beta_p, y_p, z_{pp_0})$ is complemented by the Banach subspace $\{0\} \oplus p\mathfrak{M}_*p \oplus \{0\} \oplus \{0\}$. Thus we conclude that the momentum map $J_1 : T_*P_0 \rightarrow p_0\mathfrak{M}_*p_0$ is a surjective submersion.

For $F, G \in C^\infty(p_0\mathfrak{M}_*p_0)$ and $\beta = J_1(\varphi, \eta)$ we have

$$\begin{aligned} \{F \circ J_1, G \circ J_1\}(\varphi, \eta) &= \left\langle \frac{\partial G}{\partial \beta} \frac{\partial J_1}{\partial \eta}(\varphi, \eta), \frac{\partial F}{\partial \beta} \frac{\partial J_1}{\partial \varphi}(\varphi, \eta) \right\rangle - \left\langle \frac{\partial F}{\partial \beta} \frac{\partial J_1}{\partial \eta}(\varphi, \eta), \frac{\partial G}{\partial \beta} \frac{\partial J_1}{\partial \varphi}(\varphi, \eta) \right\rangle = \\ &= \left\langle \frac{\partial F}{\partial \beta}(\beta)(\cdot)\eta, \frac{\partial G}{\partial \beta}(\beta)\varphi \right\rangle - \left\langle \frac{\partial G}{\partial \beta}(\beta)(\cdot)\eta, \frac{\partial F}{\partial \beta}(\beta)\varphi \right\rangle = \\ &= \left\langle \left[\frac{\partial F}{\partial \beta}(\beta), \frac{\partial G}{\partial \beta}(\beta) \right], \varphi\eta \right\rangle = (\{F, G\}_{LP} \circ J_1)(\varphi, \eta). \end{aligned} \quad (4.58)$$

We note here that $\frac{\partial F \circ J_1}{\partial \eta}(\varphi, \eta) = \frac{\partial F}{\partial \beta}(\beta)\varphi$ and $\frac{\partial G \circ J_1}{\partial \eta}(\varphi, \eta) = \frac{\partial G}{\partial \beta}(\beta)\varphi$ belong to $p_0\mathfrak{M}_*$. From (4.58) we conclude that J_1 is a Poisson map.

(ii) The polarity of Poisson subalgebras $J_1^*(C^\infty(p_0\mathfrak{M}_*p_0))$ and $Q_0^*(\mathcal{P}^\infty(T_*P_0/G_0))$ follows from the fact that one has $\xi^x(f) = 0$ for the vector field ξ^x defined in (4.19) and $f \in C_{G_0}^\infty(T_*P_0/G_0)$. \square

Example 4.1. As an example let us consider the case when \mathfrak{M} is a finite W^* -algebra and the projection p_0 is the unit element of \mathfrak{M} . Then P_0 is equal to the group $G(\mathfrak{M})$ of the invertible elements of \mathfrak{M} . In this case Q_0 and J_1 are the left $J_L : T_*G(\mathfrak{M}) \rightarrow \mathfrak{M}_*$ and right $J_R : T_*G(\mathfrak{M}) \rightarrow$

\mathfrak{M}_* momentum maps, respectively, of the weak symplectic manifold T_*G_0 . From Theorem 4.4 it follows that

$$\begin{array}{ccc} T_*G(\mathfrak{M}) & & \\ \downarrow J_R & \downarrow J_L & \\ \mathfrak{M}_* & & \end{array} \quad (4.59)$$

is the precotangent weak symplectic groupoid of the Banach Lie Poisson space $(\mathfrak{M}_*, \{\cdot, \cdot\}_{LP})$ with Lie-Poisson bracket $\{\cdot, \cdot\}_{LP}$ defined in (4.52).

In order to define the sub Poisson structure on $p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$ let us firstly introduce such kind of structure on $p_0\mathfrak{M}_*p_0 \times P_0$. For this reason we take the subalgebra of smooth functions

$$\mathcal{P}_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0) := \left\{ F \in C^\infty(p_0\mathfrak{M}_*p_0 \times P_0) : \frac{\partial F}{\partial \eta}(\beta, \eta) \in p_0\mathfrak{M}_* \text{ and } F(Ad_g^*\beta, \eta g) = F(\beta, \eta) \right\} \quad (4.60)$$

and the Poisson bracket of $F, G \in \mathcal{P}_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0)$ we define by

$$\{F, G\}_{sP}(\beta, \eta) := \left\langle \beta, \left[\frac{\partial F}{\partial \beta}(\beta, \eta), \frac{\partial G}{\partial \beta}(\beta, \eta) \right] \right\rangle. \quad (4.61)$$

One can easily check that $\{F, G\}_{sP} \in \mathcal{P}_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0)$. Considering $\mathcal{P}_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0)$ as the subalgebra $\mathcal{P}^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}} P_0)$ of $C^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}} P_0)$ and taking into account that the bracket (4.61) is G_0 -invariant we find that it defines a sub Poisson structure on $p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}} P_0$. Note here that $\mathcal{P}^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}} P_0) \subsetneq C^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}} P_0)$ in general.

Let us mention that the bracket (4.61) is also well define if $F, G \in C_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0)$. However, we have assumed in (4.60) the condition $\frac{\partial F}{\partial \eta}(\beta, \eta) \in p_0\mathfrak{M}_*$ for the consistency with the sub Poisson structure on T_*P_0 defined by (4.23) and (4.24).

The next proposition describe the sub Poisson structure of the predual Atiyah sequence (4.10)

Theorem 4.5. *The predual Atiyah sequence (4.10) is a short exact sequence of the fibre-wise linear sub Poisson complex Banach vector bundles, i.e.*

- (i) *The Banach vector bundle map $\iota_* : T_*P_0/G_0 \rightarrow p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$ is a sub Poisson submersion.*
- (ii) *One has $\ker \iota_* = J_1^{-1}(0)/G_0$, where $J_1^{-1}(0)/G_0$ is the weak symplectic leaf in T_*P_0/G_0 obtained by the Marsden-Weinstein symplectic reduction procedure, [15]. The predual anchor map $a_* : T_*(P_0/G_0) \hookrightarrow T_*P_0/G_0$ is an immersion which defines the isomorphism $T_*(P_0/G_0) \cong J_1^{-1}(0)/G_0$ of weak symplectic manifolds, if the precotangent bundle $T_*(P_0/G_0)$ is endowed with the canonical weak symplectic structure.*

Proof. (i) In order to describe $\iota_* : T_*P_0/G_0 \rightarrow p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$, see (4.10), in detail, we consider the map

$$I_* : T_*P_0 \cong p_0\mathfrak{M}_* \times P_0 \ni (\varphi, \eta) \mapsto (J_1(\varphi, \eta), \eta) \in p_0\mathfrak{M}_*p_0 \times P_0. \quad (4.62)$$

Note that $I_* = J_1 \times pr_2$, where $pr_2 : p_0\mathfrak{M}_* \times P_0$ is the projection on the second component of the Cartesian product. Since J_1 and pr_2 are surjective submersion we conclude that I_* has the same property, see 5.9.3 in [4].

The equivariance property $I_*(g^{-1}\varphi, \eta g) = (Ad_{g^{-1}}^*(J_1(\varphi, \eta), \eta g)$, $g \in G_0$ allows us to define ι_* by

$$\iota_*([\varphi, \eta]) := [(J_1(\varphi, \eta), \eta)] = [(\varphi, \eta)], \quad (4.63)$$

where $[(\varphi, \eta)]$ and $[(\beta, \eta)]$ are the G_0 -orbits of $(\varphi, \eta) \in p_0\mathfrak{M}_* \times P_0$ and $(\beta, \eta) \in p_0\mathfrak{M}_*p_0 \times P_0$, respectively.

In order to show that the bundle epimorphism $\iota_* : T_*P_0/P_0 \rightarrow p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$ is a submersion we note that

$$\iota_* \circ Q_0 = \pi_{Ad_{G_0}^*} \circ I_*,$$

where $\pi_{Ad_{G_0}^*} : p_0\mathfrak{M}_*p_0 \times P_0 \rightarrow p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$ is the quotient map. Since the maps Q_0 , $\pi_{Ad_{G_0}^*}$ and I_* are surjective submersions we conclude that ι_* is a submersion too, see 5.9.2 in [4].

For $F, G \in \mathcal{P}_{G_0}^\infty(p_0\mathfrak{M}_*p_0 \times P_0) \cong \mathcal{P}^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0)$ we have

$$\begin{aligned} \{F \circ I_*, G \circ I_*\}(\varphi, \eta) &= \\ &= \left\langle \frac{\partial(G \circ I_*)}{\partial\eta}(\varphi, \eta), \frac{\partial(F \circ I_*)}{\partial\varphi}(\varphi, \eta) \right\rangle - \left\langle \frac{\partial(F \circ I_*)}{\partial\eta}(\varphi, \eta), \frac{\partial(G \circ I_*)}{\partial\varphi}(\varphi, \eta) \right\rangle = \\ &= \left\langle \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta) \frac{\partial J_1}{\partial\eta}(\varphi, \eta) + \frac{\partial G}{\partial\eta}(J_1(\varphi, \eta), \eta), \frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta)(\cdot)\eta \right\rangle + \\ &\quad - \left\langle \frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta) \frac{\partial J_1}{\partial\eta}(\varphi, \eta) + \frac{\partial F}{\partial\eta}(J_1(\varphi, \eta), \eta), \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta)(\cdot)\eta \right\rangle = \\ &= \left\langle \frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta), \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta)\varphi\eta + \frac{\partial G}{\partial\eta}(J_1(\varphi, \eta), \eta)\eta \right\rangle + \\ &\quad - \left\langle \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta), \frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta)\varphi\eta + \frac{\partial F}{\partial\eta}(J_1(\varphi, \eta), \eta)\eta \right\rangle = \\ &= \left\langle J_1(\varphi, \eta), \left[\frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta), \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta) \right] \right\rangle + \\ &\quad + \left\langle \frac{\partial G}{\partial\eta}(J_1(\varphi, \eta), \eta), \eta \frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta) \right\rangle - \left\langle \frac{\partial F}{\partial\eta}(J_1(\varphi, \eta), \eta), \eta \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta) \right\rangle \end{aligned} \quad (4.64)$$

Since the functions F and G are G_0 -invariant then

$$\left\langle \frac{\partial G}{\partial\eta}(\beta, \eta), \eta x \right\rangle = - \left\langle \frac{\partial G}{\partial\beta}(\beta, \eta), ad_x^*\beta \right\rangle = - \left\langle \left[x, \frac{\partial G}{\partial\beta}(\beta, \eta) \right], \beta \right\rangle \quad (4.65)$$

for any $x \in p_0\mathfrak{M}_*p_0$.

Thus, taking $x = \frac{\partial F}{\partial\beta}(\beta, \eta)$ and $x = \frac{\partial G}{\partial\beta}(\beta, \eta)$, respectively, we obtain

$$\begin{aligned} \{F \circ I_*, G \circ I_*\}(\varphi, \eta) &= \left\langle J_1(\varphi, \eta), \left[\frac{\partial F}{\partial\beta}(J_1(\varphi, \eta), \eta), \frac{\partial G}{\partial\beta}(J_1(\varphi, \eta), \eta) \right] \right\rangle = \\ &= (\{F, G\}_{LP} \circ J_1)(\varphi, \eta) = (\{F, G\}_{sP} \circ I_*)(\varphi, \eta). \end{aligned} \quad (4.66)$$

Since both Poisson brackets in (4.66) and functions F, G are G_0 -invariant one can take the quotient of (4.66) by G_0 . Hence we obtain

$$\{F \circ \iota_*, G \circ \iota_*\} = (\{F, G\}_{sP} \circ \iota_*), \quad (4.67)$$

where $\{F, G\}_{sP}$ is the Poisson bracket on $\mathcal{P}^\infty(p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0)$ defined by the Poisson bracket (4.61).

- (ii) Since the momentum map $J_1 : T_*P_0 \rightarrow p_0\mathfrak{M}_*p_0$ is a submersion the fibre $J_1^{-1}(0)$ of $0 \in p_0\mathfrak{M}_*p_0$ is a submanifold of T_*P_0 . So, $J_1^{-1}(0)/G_0$ is a submanifold of T_*P_0/G_0 .

The equality $\ker \iota_* = J_1^{-1}(0)/G_0$ follows directly from (4.63). In order to show that the Marsden-Weinstein symplectic reduction applied to $J_1^{-1}(0)$ leads to the weak symplectic manifold structure on $J_1^{-1}(0)/G_0$ we define the local trivialization $a_{*p} : \nu_*^{-1}(\Pi_p) \rightarrow (\pi_* \circ \pi_0)^{-1}(\Pi_p)$ of the predual anchor map a_* , where $\nu_* : T_*(P_0/G_0) \rightarrow P_0/G_0$ is the bundle projection of the precotangent bundle $T_*(P_0/G_0)$.

For any $p \in \mathcal{L}_{p_0}(\mathfrak{M}) \cong P_0/G_0$ we choose $\eta_{pp_0} \in P_0$ such that $\mathbf{t}(\eta_{pp_0}) = p$ and define the principal bundle section $\sigma_{pp_0} : \Pi_p \rightarrow \pi_0^{-1}(\Pi_p) \subset P_0$ by

$$\sigma_{pp_0}(q) := \sigma_p(q)\eta_{pp_0}, \quad q \in \Pi_p. \quad (4.68)$$

Using σ_{pp_0} we define a_{*p} as follows

$$a_{*p}(\rho) := (T\pi_0(\sigma_{pp_0}(q)))^*(\rho) = (\rho \circ T\pi_0)(\sigma_{pp_0}(q)), \quad (4.69)$$

where $\rho \in \nu_*^{-1}(q)$.

For $\xi_\rho \in T_\rho(T_*(P_0/G_0))$ and the pullback $(a_{*p})^*\gamma$ of the canonical 1-form (4.16) we have

$$\begin{aligned} \langle ((a_{*p})^*\gamma)_\rho, \xi_\rho \rangle &= \langle \gamma_{a_{*p}(\rho)}, Ta_{*p}(\rho)\xi_\rho \rangle = \\ &= \langle a_{*p}(\rho), T\pi_*(a_{*p}(\rho))Ta_{*p}(\rho)\xi_\rho \rangle = \langle a_{*p}(\rho), T(\pi_* \circ a_{*p})(\rho)\xi_\rho \rangle = \langle \rho, T\pi_0(\sigma_{pp_0}(q)) \circ T(\pi_* \circ a_{*p})(\rho)\xi_\rho \rangle = \\ &= \langle \rho, T(\pi_0 \circ \pi_* \circ a_{*p})(\rho)\xi_\rho \rangle = \langle \rho, T\nu_*(\rho)\xi_\rho \rangle =: \langle \tilde{\gamma}_\rho, \xi_\rho \rangle \end{aligned} \quad (4.70)$$

The last equality in (4.70) follows from $\pi_0 \circ \pi_* \circ a_{*p} = \nu_*$. From (4.70) we conclude that the pullback $(a_{*p})^*\gamma$ does not depend on the trivialization and is equal to the weak canonical form $\tilde{\gamma}$ of $T_*(P_0/G_0)$. It also follows from (4.70) that $a_* : T_*(P_0/G_0) \xrightarrow{\sim} J_1^{-1}(0)/G_0$ is an isomorphism of weak symplectic manifolds. \square

Corollary 4.6. *All statement of Theorem 4.5 are valid for (4.11). Thus, since of Proposition 1.2 they are also valid for (4.2).*

Proof. It follows from Proposition 4.1. \square

The bracket (4.23) written in the local coordinates (4.43) assumes the following form

$$\begin{aligned} \{f, g\} &= \left\langle \frac{\partial g}{\partial y_p}, \frac{\partial f}{\partial \alpha_p} \right\rangle - \left\langle \frac{\partial f}{\partial y_p}, \frac{\partial g}{\partial \alpha_p} \right\rangle + \\ &+ \left\langle \beta_p, \left[\frac{\partial g}{\partial \beta_p}, \frac{\partial f}{\partial \beta_p} \right] \right\rangle + \left\langle z_{pp_0} \frac{\partial g}{\partial z_{pp_0}}, \frac{\partial f}{\partial \beta_p} \right\rangle - \left\langle z_{pp_0} \frac{\partial f}{\partial z_{pp_0}}, \frac{\partial g}{\partial \beta_p} \right\rangle. \end{aligned} \quad (4.71)$$

We note here that for $f \in \mathcal{P}^\infty(T_*P_0)$, i.e. $\frac{\partial f}{\partial \eta}(\eta) \in p_0\mathfrak{M}_*$, the partial derivative $\frac{\partial f}{\partial y_p}(\alpha_p, \beta_p, y_p, z_{pp_0})$ belongs to $p\mathfrak{M}_*(1-p)$. The sub Poisson anchor $\#_1 : T^\flat(T_*P_0) \rightarrow T(T_*P_0)$ in these coordinates is written as

$$\#_1(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{y}_p, \overset{\circ}{z}_{pp_0}, \alpha_p, \beta_p, y_p, z_{pp_0}) = (-\overset{\circ}{y}_p, -ad_{\beta_p}^*(\beta_p) - z_{pp_0} \overset{\circ}{z}_{pp_0}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, z_{pp_0}, \alpha_p, \beta_p, y_p, z_{pp_0}) \quad (4.72)$$

If $f, g \in \mathcal{P}_{G_0}^\infty(T_*P_0) \cong \mathcal{P}^\infty(T_*P_0/G_0)$ then $z_{pp_0} \frac{\partial g}{\partial z_{pp_0}} = 0$ and $z_{pp_0} \frac{\partial f}{\partial z_{pp_0}} = 0$. Hence the two last terms in (4.71) disappear and one obtains the local coordinate formula

$$\{f, g\}_{G_0} = \left\langle \frac{\partial g}{\partial y_p}, \frac{\partial f}{\partial \alpha_p} \right\rangle - \left\langle \frac{\partial f}{\partial y_p}, \frac{\partial g}{\partial \alpha_p} \right\rangle + \left\langle \beta_p, \left[\frac{\partial g}{\partial \beta_p}, \frac{\partial f}{\partial \beta_p} \right] \right\rangle \quad (4.73)$$

for the Poisson bracket $\{\cdot, \cdot\}_{G_0}$ on T_*P_0/G_0 . The sub Poisson anchor $\#_{1G_0} : T^\flat(T_*P_0/G_0) \rightarrow T(T_*P_0/G_0)$ according to (4.73) is as follows

$$\#_{1G_0}(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{y}_p, \alpha_p, \beta_p, y_p) = (-\overset{\circ}{y}_p, -ad_{\overset{\circ}{\beta}_p}^*(\beta_p), \overset{\circ}{\alpha}_p, \alpha_p, \beta_p, y_p), \quad (4.74)$$

where $(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{y}_p) \in (1-p)\mathfrak{M}_p \times p\mathfrak{M}_p \times p\mathfrak{M}_*(1-p)$ are the coordinates along fibres of $T^\flat(T_*P_0/G_0) \rightarrow T_*P_0/G_0$.

Therefore, the coordinates (y_p, α_p, β_p) are the canonical coordinates in sense of Weinstein local splitting theorem presented in [22], see also formulas (1.41), (1.42) and (1.43) in Chapter 1 of [9]. Hence, one can consider the predual Atiyah sequence (4.10) as a global version of the local splitting theorem for the sub Poisson of the complex Banach manifold T_*P_0/G_0 .

Using the G_0 -invariance of (α_p, β_p, y_p) one easily concludes that (α_p, y_p) are local coordinates on $T_*(P_0/G_0)$ and (β_p, y_p) on $p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$. The canonical Poisson bracket $\{\cdot, \cdot\}$ on $T_*(P_0/G_0)$ and the Poisson bracket $\{\cdot, \cdot\}_{sP}$ on $p_0\mathfrak{M}_*p_0 \times_{Ad_{G_0}^*} P_0$ written in the above coordinates are given by appropriate parts of (4.73) if we substitute to it the functions f and g dependent only on (α_p, y_p) and (β_p, y_p) , respectively.

The coordinate formula for the sub Poisson anchor $\tilde{\#}_1 : T^\flat(T_*(P_0/G_0)) \rightarrow T(T_*(P_0/G_0))$ of the weak symplectic manifold $T^*(T_*(P_0/G_0))$ is as follows

$$\tilde{\#}_1(\overset{\circ}{\alpha}_p, \overset{\circ}{y}_p, \alpha_p, y_p) = (-\overset{\circ}{y}_p, \overset{\circ}{\alpha}_p, \alpha_p, y_p). \quad (4.75)$$

In these coordinates the predual anchor map a_* is given by $(\alpha_p, y_p) \mapsto (\alpha_p, 0, y_p)$ and the map ι_* is given by $(\alpha_p, \beta_p, y_p) \mapsto (\beta_p, y_p)$. These observations taken together show again that the predual Atiyah sequence (4.10) is a short exact sequence of sub Poisson Banach bundles.

As we see from Theorem 4.4 the weak symplectic realization (4.53) of sub Poisson manifold T_*P_0/G_0 and the Banach-Lie Poisson space $p_0\mathfrak{M}_*p_0$ gives the correspondence between their symplectic leaves. Namely, a coadjoint orbit $\mathcal{O} \subset p_0\mathfrak{M}_*p_0$, which is a weak symplectic leave in the Banach-Lie Poisson space $p_0\mathfrak{M}_*p_0$, corresponds to the syplectic leave

$$\pi_{*G_0}(J_1^{-1}(0)) = \iota_*^{-1}(\mathcal{O} \times_{Ad_{G_0}^*} P_0) \quad (4.76)$$

in T_*P_0/G_0 .

Let us mention that the symplectic leaves of Banach-Lie Poisson spaces were investigated in [2, 3, 16]. For example, in the case of $L^1(\mathcal{H})$, which is predual of $L^\infty(\mathcal{H})$, the coadjoint orbit \mathcal{O}_ρ of the finite rank trace class operator $\rho \in L^1(\mathcal{H})$ is a submanifold of $L^1(\mathcal{H})$ and its symplectic structure is given by the strong symplectic form. However, the investigation of the symplectic leaves of \mathfrak{M}_* , and thus T_*P_0/G_0 , needs the advanced functional analytical methods and is not easy even in a concrete case.

5 Predual short exact sequence of \mathcal{VB} -groupoids with $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ as the side groupoid

Through this section we will study the sub Poisson structure of some Banach-Lie \mathcal{VB} -groupoids which have the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ as the side groupoid, see diagram (5.20). An

introduction to the theory of \mathcal{VB} -groupoids can be found in [13]. Indispensable ingredients of this theory are also presented in [14] and in the appendix of this paper.

Applying the tangent functor to a Banach Lie groupoid $G \rightrightarrows M$ one obtains its tangent \mathcal{VB} -groupoid $TG \rightrightarrows TM$. In particular case, one obtains the tangent group $TG \rightrightarrows \{0\}$ of a Banach Lie group $G \rightrightarrows \{e\}$. The tangent groupoid $TG \rightrightarrows TM$ as well as its dual $T^*G \rightrightarrows A^*G$, where AG is the algebroid of the groupoid G , yield important examples of \mathcal{VB} -groupoids.

We modify the definition of the finite dimensional Poisson groupoid presented in Chapter 11.4 of [13] to the sub Poisson Banach case considered here. According to this modification the Banach-Lie groupoid $G \rightrightarrows M$ is a **sub Poisson groupoid** with a sub Poisson anchor $\# : T^bG \rightarrow TG$ if there exists a Banach subgroupoid $T^bG \rightrightarrows A^bG$ of the Banach groupoid $T^*G \rightrightarrows A^*G$ dual to $TG \rightrightarrows TM$ and Banach bundles morphism $a_* : A^bG \rightarrow TM$ such that

$$\begin{array}{ccc} T^bG & \xrightarrow{\#} & TG \\ \downarrow \downarrow & & \downarrow \downarrow \\ A^bG & \xrightarrow{a_*} & TM \end{array} \quad (5.1)$$

is a morphism of \mathcal{VB} -groupoids, where by AG we have denoted the algebroid of $G \rightrightarrows P$.

Our considerations we begin observing that Atiyah sequences (2.13) and (2.16) could be incorporated in the short exact sequence of \mathcal{VB} -groupoids

$$\begin{array}{ccccccc} p_0\mathfrak{M}_{p_0 \times Ad_{G_0}}(P_0 \times P_0) & \xrightarrow{\quad} & \frac{P_0 \times P_0}{G_0} & \xrightarrow{\quad} & \frac{P_0 \times P_0}{G_0} & \xrightarrow{\quad} & \frac{P_0 \times P_0}{G_0} \\ \downarrow \downarrow & \searrow \iota_2 & \downarrow \downarrow & \searrow a_2 & \downarrow \downarrow & \searrow & \downarrow \downarrow \\ p_0\mathfrak{M}_{p_0 \times Ad_{G_0}}P_0 & \xrightarrow{\quad} & \frac{P_0}{G_0} & \xrightarrow{\quad} & T\left(\frac{P_0 \times P_0}{G_0}\right) & \xrightarrow{\quad} & \frac{P_0 \times P_0}{G_0} \\ \downarrow \downarrow & \searrow \iota & \downarrow \downarrow & \searrow a & \downarrow \downarrow & \searrow & \downarrow \downarrow \\ & & TP_0/G_0 & \xrightarrow{\quad} & \frac{P_0}{G_0} & \xrightarrow{\quad} & \frac{P_0}{G_0} \\ & & \downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\ & & T(P_0/G_0) & \xrightarrow{\quad} & T(P_0/G_0) & \xrightarrow{\quad} & \frac{P_0}{G_0} \end{array} \quad (5.2)$$

which have the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ as their common side groupoid, where the vertical arrows in (5.2) are the respective source and target maps.

Let us shortly describe the \mathcal{VB} -groupoids included in the short exact sequence (5.2).

- (i) The structural maps of the left hand side \mathcal{VB} -groupoid in (5.2) are defined as follows

$$\begin{aligned} \tilde{s}(\langle x, \eta, \xi \rangle) &:= \langle x, \xi \rangle, \\ \tilde{t}(\langle x, \eta, \xi \rangle) &:= \langle x, \eta \rangle, \\ \tilde{\mathbf{I}}(\langle x, \eta \rangle) &= \langle x, \eta, \eta \rangle, \\ \tilde{0}(\langle \eta, \xi \rangle) &= \langle 0, \eta, \xi \rangle, \end{aligned}$$

The groupoid product of the elements $\langle x, \eta, \xi \rangle, \langle y, \zeta, \delta \rangle \in p_0 \mathfrak{M} p_0 \times_{Ad_{G_0}} (P_0 \times P_0)$ such that $\tilde{s}(\langle x, \eta, \xi \rangle) = \tilde{t}(\langle y, \zeta, \delta \rangle)$, what means that $\langle x, \xi \rangle = \langle y, \zeta \rangle$, is defined by

$$\langle x, \eta, \xi \rangle \langle y, \zeta, \delta \rangle = \langle x, \eta, \delta g^{-1} \rangle, \quad (5.3)$$

where $g \in G_0$ satisfies $\zeta = \xi g$ and $y = Ad_g x$. The groupoid inverse map is

$$\tilde{l}(\langle x, \eta, \xi \rangle) := \langle x, \xi, \eta \rangle.$$

(ii) The central \mathcal{VB} -groupoid in (5.2) is the quotient by G_0 of the tangent groupoid $TP_0 \times TP_0 \rightrightarrows TP_0$ of the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$.

(iii) The right-hand side \mathcal{VB} -groupoid in (5.2) is the tangent groupoid of $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$.

We see that the short exact sequence (5.2) involves various fundamental structures, i.e. the vector bundle, the principal bundle, the groupoid and the algebroid structures, which are consistently related one with another.

Let us now apply the dualization procedure, discussed in the subsections 7.2 and 7.3 of the Appendix, to (5.2). For this reason we observe that as in the case of Atiyah sequences (2.13) and (2.16) one can define (5.2) as the quotient of

$$\begin{array}{ccccccc} p_0 \mathfrak{M} p_0 \times (P_0 \times P_0) & \xrightarrow{\quad} & P_0 \times P_0 & & & & \\ \parallel & \searrow I_2 & \parallel & \searrow & & & \\ p_0 \mathfrak{M} p_0 \times P_0 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & TP_0 \times TP_0 & \xrightarrow{\quad} & P_0 \times P_0 \\ & \searrow I & \parallel & \searrow A_2 & \parallel & \searrow & \\ & & TP_0 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & \frac{T(P_0 \times P_0)}{p_0 \mathfrak{M} p_0} \xrightarrow{\quad} P_0 \times P_0 \\ & & & \searrow A & \parallel & \searrow & \\ & & & & TP_0/p_0 \mathfrak{M} p_0 & \xrightarrow{\quad} & P_0 \end{array} \quad (5.4)$$

by G_0 .

Proposition 5.1. *The cores of Banach \mathcal{VB} -groupoids included in the short exact sequence (5.4) are:*

$$core(p_0 \mathfrak{M} p_0 \times P_0 \times P_0 \rightrightarrows p_0 \mathfrak{M} p_0 \times P_0) \cong P_0 \times \{0\}, \quad (5.5)$$

$$core(TP_0 \times TP_0 \rightrightarrows TP_0) \cong TP_0 \cong \mathfrak{M} p_0 \times P_0, \quad (5.6)$$

$$core\left(\frac{T(P_0 \times P_0)}{p_0 \mathfrak{M} p_0} \rightrightarrows TP_0/p_0 \mathfrak{M} p_0\right) \cong TP_0 \cong \mathfrak{M} p_0 \times P_0. \quad (5.7)$$

Proof. From the definition (7.2) in the appendix we see that $(\eta, x, \xi) \in core(p_0 \mathfrak{M} p_0 \times P_0 \times P_0 \rightrightarrows p_0 \mathfrak{M} p_0 \times P_0)$ if and only if $\eta = \xi$ and $x = 0$. Thus one has (5.5).

The element $(v, \eta, w, \xi) \in core(TP_0 \times TP_0 \rightrightarrows TP_0)$ if and only if $\eta = \xi$ and $(w, \xi) = (0, \xi)$. So, we have $core(TP_0 \times TP_0 \rightrightarrows TP_0) = \{(v, \eta, 0, \eta) \in TP_0 \times TP_0\} \cong TP_0$.

Any element $\langle v, \eta, w, \xi \rangle \in \frac{TP_0 \times TP_0}{p_0 \mathfrak{M} p_0}$ is defined by

$$\langle v, \eta, w, \xi \rangle := \{(v + \eta x, \eta, w + \xi x, \xi); \quad x \in p_0 \mathfrak{M} p_0\}. \quad (5.8)$$

Then $\langle v, \eta, w, \xi \rangle \in \text{core} \left(\frac{T(P_0 \times P_0)}{p_0 \mathfrak{M} p_0} \rightrightarrows TP_0/p_0 \mathfrak{M} p_0 \right)$ if and only if $\eta = \xi$ and $\langle w, \xi \rangle = \langle 0, \xi \rangle$. Thus $w = \xi y$ for some $y \in p_0 \mathfrak{M} p_0$. If $x = -y$ then we find that

$$\text{core} \left(\frac{TP_0 \times TP_0}{p_0 \mathfrak{M} p_0} \rightrightarrows TP_0/p_0 \mathfrak{M} p_0 \right) = \{ \langle v, \eta, 0, \eta \rangle; \quad (v, \eta) \in TP_0 \} \cong TP_0.$$

□

Using isomorphisms from Proposition 5.1 and applying the dualization procedure described in the subsection 7.3 of the Appendix, to (5.4) we obtain the short exact sequence of Banach \mathcal{VB} -groupoids

$$\begin{array}{ccccccc} T^0(P_0 \times P_0) & \longrightarrow & P_0 \times P_0 & & & & \\ \parallel & & \parallel & \searrow & A_2^* & & \\ & & T^*P_0 \times T^*P_0 & \longrightarrow & P_0 \times P_0 & & \\ \parallel & & \parallel & & \parallel & \searrow & \\ T^*P_0 & \longrightarrow & P_0 & & I_2^* & & (p_0 \mathfrak{M} p_0)^* \times P_0 \times P_0 \longrightarrow P_0 \times P_0 \\ & & \parallel & & \parallel & & \parallel \\ & & T^*P_0 & \longrightarrow & P_0 & & \\ & & & & \parallel & & \\ & & & & \{0\} \times P_0 & \longrightarrow & P_0 \end{array} \quad (5.9)$$

dual to (5.4). The Banach vector subbundle $T^0(P_0 \times P_0) \rightarrow P_0 \times P_0$ of the Banach vector bundle $T^*(P_0 \times P_0) \rightarrow P_0 \times P_0$ consists of such covectors which annihilate $I_2(p_0 \mathfrak{M} p_0 \times P_0 \times P_0)$, i.e. by the definition one has

$$T^0(P_0 \times P_0) := \{(\varphi, \eta, \psi, \xi) \in T_*P_0 \times T_*P_0 : \quad \varphi\eta + \psi\xi = 0\} = J_2^{-1}(0), \quad (5.10)$$

where

$$J_2(\varphi, \eta, \psi, \xi) = \varphi\eta + \psi\xi \quad (5.11)$$

is the momentum map for weak symplectic manifold $T_*(P_0 \times P_0) \cong T_*P_0 \times T_*P_0$. The bundle monomorphism A_2^* dual to A_2 is an inclusion map and the bundle epimorphism I_2^* dual to I_2 is given by

$$I_2^*(\varphi, \eta, \psi, \xi) := (\varphi\eta + \psi\xi, \eta, \xi) \quad (5.12)$$

The structural maps of a \mathcal{VB} -groupoid, see the subsection 7.1 of the Appendix, in the case of

$$\begin{array}{ccc} T^*P_0 \times T^*P_0 & \longrightarrow & P_0 \times P_0 \\ \parallel & & \parallel \\ T^*P_0 & \longrightarrow & P_0 \end{array}, \quad (5.13)$$

are the following:

$$\begin{aligned}
\tilde{\mathbf{s}}_*(\varphi, \eta, \psi, \xi) &= (-\psi, \xi), \\
\tilde{\mathbf{t}}_*(\varphi, \eta, \psi, \xi) &= (\varphi, \eta), \\
\tilde{\mathbf{l}}_*(\varphi, \eta) &= (\varphi, \eta, -\varphi, \eta), \\
\tilde{\lambda}_*(\varphi, \eta, \psi, \xi) &= (\eta, \xi), \\
\tilde{0}_*(\eta, \xi) &= (0, \eta, 0, \xi), \\
\lambda_*(\varphi, \eta) &= \eta, \\
0_*(\eta) &= (0, \eta).
\end{aligned} \tag{5.14}$$

Its inverse map and the groupoid product are given by

$$\iota_*(\varphi, \eta, \psi, \xi) = (-\psi, \xi, -\varphi, \eta), \tag{5.15}$$

and

$$(\varphi, \eta, \psi, \xi)(-\psi, \xi, \lambda, \zeta) = (\varphi, \eta, \lambda, \zeta),$$

respectively.

The left hand side Banach \mathcal{VB} -groupoid $T^0(P_0 \times P_0) \rightrightarrows T^*P_0$ in (5.9) is a Banach subgroupoid of the intermediate \mathcal{VB} -groupoid in (5.9). So, its structure is defined by (5.14) and (5.15).

The structure of the right hand side of (5.9) Banach \mathcal{VB} -groupoid

$$\begin{array}{ccc}
p_0\mathfrak{M}_{*p_0} \times P_0 \times P_0 & \longrightarrow & P_0 \times P_0 \\
\downarrow \downarrow & & \downarrow \downarrow \\
\{0\}^* \times P_0 & \longrightarrow & P_0
\end{array} \tag{5.16}$$

is given by:

$$\begin{aligned}
\tilde{\mathbf{s}}_*(\mathcal{X}, \eta, \xi) &:= (0, \xi), \\
\tilde{\mathbf{t}}_*(\mathcal{X}, \eta, \xi) &:= (0, \eta), \\
\tilde{\mathbf{l}}_*(0, \eta) &:= (0, \eta, \eta), \\
\tilde{\lambda}_*(\mathcal{X}, \eta, \xi) &= (\eta, \xi), \\
\tilde{0}_*(\eta, \xi) &= (0, \eta, \xi), \\
\lambda_*(0, \eta) &= \eta, \\
0_*(\eta) &= (0, \eta)
\end{aligned} \tag{5.17}$$

and by

$$\begin{aligned}
\iota_*(\mathcal{X}, \eta, \xi) &:= (-\mathcal{X}, \eta, \xi), \\
(\mathcal{X}, \eta, \xi)(\mathcal{Y}, \xi, \zeta) &:= (\mathcal{X} + \mathcal{Y}, \eta, \zeta),
\end{aligned} \tag{5.18}$$

where in (5.87) the structural maps and in (5.88) the product and inverse map are defined. We pay attention here to the fact that the map $(\varphi, \eta, \psi, \xi) \mapsto (\psi, \xi, -\varphi, \eta)$ defines an isomorphism of the Banach \mathcal{VB} -groupoid (5.13) with the pair \mathcal{VB} -groupoid $T^*P_0 \times T^*P_0 \rightrightarrows T^*P_0$. However, the distinction between these \mathcal{VB} -groupoids is crucial for further investigations.

Remark 5.2. Replacing in (5.9) T^*P_0 by T_*P_0 and $(p_0\mathfrak{M}_{*p_0})^*$ by $p_0\mathfrak{M}_{*p_0} \cong (p_0\mathfrak{M}_{*p_0})_*$ one obtains the short exact sequence of Banach \mathcal{VB} -groupoids

$$\begin{array}{ccccccc}
T^0(P_0 \times P_0) & \xrightarrow{\quad} & P_0 \times P_0 & & & & \\
\parallel & \searrow A_2^* & \parallel & \searrow & & & \\
T_*P_0 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & T_*P_0 & \xrightarrow{\quad} & P_0 \\
& & \parallel & & \parallel & & \parallel \\
& & T_*P_0 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & P_0 \\
& & & & \parallel & & \parallel \\
& & & & \{0\} \times P_0 & \xrightarrow{\quad} & P_0
\end{array}
\quad (5.19)$$

We recall here that $T^*P_0 \cong (p_0\mathfrak{M}p_0)^* \times P_0$ and $T_*P_0 \cong (p_0\mathfrak{M}p_0)_* \times P_0$ and all morphisms, structural maps and groupoid operations in (5.9) respect canonical inclusions $(p_0\mathfrak{M}p_0)_* \subset (p_0\mathfrak{M}p_0)^*$ and $(\mathfrak{M}p_0)_* \subset (\mathfrak{M}p_0)^*$ of the Banach spaces.

Remark 5.3. The Banach bundles of (5.19) are only the quasi Banach subbundles of their counterparts in (5.9). Because the predual Banach spaces $p_0\mathfrak{M}_*p_0$ and $p_0\mathfrak{M}_*$ do not have respective Banach complements in $(p_0\mathfrak{M}p_0)^*$ and $(\mathfrak{M}p_0)^*$.

Let us note that the epimorphism I_2^* , as well as the inclusion A_2^* , have the G_0 -equivariance property, i.e.

$$I_2^*(g^{-1}\varphi, \eta g, g^{-1}\psi, \xi g) = (Ad_{g^{-1}}^*(\varphi\eta + \psi\xi), \eta g, \xi g),$$

where $g \in G_0$. Hence we have

Remark 5.4. All arrows in (5.19) are equivariant with respect to G_0 . The actions of G_0 on (5.9) and (5.19) are free and quotient maps defined by them are surjective submersions.

Taking into account Remark 5.2 and Remark 5.4, and then quotienting (5.19) by G_0 we obtain the following short exact sequence of Banach \mathcal{VB} -groupoids:

$$\begin{array}{ccccccc}
T_*\left(\frac{P_0 \times P_0}{G_0}\right) & \xrightarrow{\quad} & \frac{P_0 \times P_0}{G_0} & & & & \\
\parallel & \searrow a_2^* & \parallel & \searrow & & & \\
T_*P_0/G_0 & \xrightarrow{\quad} & P_0/G_0 & \xrightarrow{\quad} & T_*P_0/G_0 & \xrightarrow{\quad} & P_0/G_0 \\
& & \parallel & & \parallel & & \parallel \\
& & T_*P_0/G_0 & \xrightarrow{\quad} & P_0/G_0 & \xrightarrow{\quad} & P_0/G_0 \\
& & & & \parallel & & \parallel \\
& & & & \{0\} \times P_0/G_0 & \xrightarrow{\quad} & P_0/G_0
\end{array}
\quad (5.20)$$

which have the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ as their side groupoid. As it was mentioned at the beginning of this section these \mathcal{VB} -groupoids will be the main object of investigations in this section.

Since the dualization procedure commute with the quotienting by G_0 we easily show that:

Remark 5.5. After application of the dualization procedure to (5.20) we come back to the short exact sequence (5.2).

Remark 5.6. The Banach spaces considered here are not reflexive in general. So, the short exact sequence of \mathcal{VB} -groupoids dual to (5.2) can not be equal to (5.20).

The upper horizontal part of (5.20)

$$\begin{array}{ccccc}
T_*\left(\frac{P_0 \times P_0}{G_0}\right) & \xrightarrow{a_2^*} & \frac{T_*P_0 \times T_*P_0}{G_0} & \xrightarrow{\ell_2^*} & \frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0} \\
\downarrow & & \downarrow & & \downarrow \\
\frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0} & \xrightarrow{\sim} & \frac{P_0 \times P_0}{G_0}
\end{array} \tag{5.21}$$

is the predual Atiyah sequence for the G_0 -principal bundle $\pi_2 : P_0 \times P_0 \rightarrow \frac{P_0 \times P_0}{G_0}$. Thus, if one defines the sub Poisson structure on $T_*P_0 \times T_*P_0$ by

$$\{f, g\} := \left\langle \frac{\partial g}{\partial \eta}, \frac{\partial f}{\partial \varphi} \right\rangle + \left\langle \frac{\partial g}{\partial \xi}, \frac{\partial f}{\partial \psi} \right\rangle - \left\langle \frac{\partial f}{\partial \eta}, \frac{\partial g}{\partial \varphi} \right\rangle - \left\langle \frac{\partial f}{\partial \xi}, \frac{\partial g}{\partial \psi} \right\rangle, \tag{5.22}$$

where $f, g \in \mathcal{P}^\infty(T_*P_0 \times T_*P_0)$, and on $p_0\mathfrak{M}_*p_0 \times P_0 \times P_0$ by

$$\{F, G\}_{sP}(\beta, \eta, \xi) := - \left\langle \beta, \left[\frac{\partial F}{\partial \beta}(\beta, \eta, \xi), \frac{\partial G}{\partial \beta}(\beta, \eta, \xi) \right] \right\rangle, \tag{5.23}$$

where $f, g \in C^\infty(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0)$, then Proposition 4.2, Remark 4.3, Theorem 4.4 and Theorem 4.5 are also valid for all ingredients of (5.21). The Poisson algebra $\P^\infty(T_*P_0 \times T_*P_0)$ in this case is defined by

$$\mathcal{P}^\infty(T_*P_0 \times T_*P_0) := \{f \in C^\infty(T_*P_0 \times T_*P_0) : \frac{\partial f}{\partial \eta}(\varphi, \eta, \psi, \xi), \frac{\partial f}{\partial \xi}(\varphi, \eta, \psi, \xi) \in p_0\mathfrak{M}_*\}. \tag{5.24}$$

Now, similarly to the previous sections we discuss the coordinate description of the sub Poisson structures pictured by the diagrams (5.19) and (5.21). For this reason we present the list of charts consistent with the fibre bundle structures of the manifolds included in these diagrams and express the respective Poisson brackets using the suitable coordinates.

- (i) On $\frac{P_0 \times P_0}{G_0} \cong \mathcal{G}_{p_0}(\mathfrak{M})$ (see (1.28) for this isomorphism) one has the coordinates $(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \in (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p}$ defined in (1.8). Note that

$$z_{p\tilde{p}} := z_{pp_0} \tilde{z}_{\tilde{p}p_0}^{-1}, \tag{5.25}$$

where (y_p, z_{pp_0}) and $(\tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0})$ are coordinates on $P_0 \cap \Omega_{pp_0}$ and $P_0 \cap \Omega_{\tilde{p}p_0}$, respectively.

- (ii) The coordinates

$$(\alpha_p, \beta_{\tilde{p}p}, \tilde{\alpha}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \in p\mathfrak{M}_*(1-p) \times \tilde{p}\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p} \tag{5.26}$$

are the canonical coordinates on $T_*\left(\frac{P_0 \times P_0}{G_0}\right)$, i.e. the variables $(\alpha_p, \beta_{\tilde{p}p}, \tilde{\alpha}_{\tilde{p}})$ are the predual to the variables $(a_p, b_{p\tilde{p}}, \tilde{a}_{\tilde{p}}) \in (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p}$. The Poisson bracket of $f, g \in \mathcal{P}^\infty\left(T_*\left(\frac{P_0 \times P_0}{G_0}\right)\right)$ defined by the canonical weak symplectic structure of $T_*\left(\frac{P_0 \times P_0}{G_0}\right)$ written in the coordinates (5.26) assumes the form

$$\{f, g\} = \left\langle \frac{\partial g}{\partial y_p}, \frac{\partial f}{\partial \alpha_p} \right\rangle - \left\langle \frac{\partial f}{\partial y_p}, \frac{\partial g}{\partial \alpha_p} \right\rangle + \tag{5.27}$$

$$+ \left\langle \frac{\partial g}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial f}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial g}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle + \left\langle \frac{\partial g}{\partial \tilde{z}_{p\tilde{p}}}, \frac{\partial f}{\partial \tilde{\beta}_{p\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial \tilde{z}_{p\tilde{p}}}, \frac{\partial g}{\partial \tilde{\beta}_{p\tilde{p}}} \right\rangle.$$

The coordinate formula of the sub Poisson anchor $\tilde{\#}_2 : T^b(T_*(\frac{P_0 \times P_0}{G_0})) \rightarrow T(T_*(\frac{P_0 \times P_0}{G_0}))$ defined by (5.27) is the following

$$\tilde{\#}_2(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}) = (-\overset{\circ}{y}_p, -\overset{\circ}{z}_{p\tilde{p}}, -\overset{\circ}{\tilde{y}}_{\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}), \quad (5.28)$$

where $(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_{p\tilde{p}}, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}) \in (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p} \times p\mathfrak{M}_*(1-p) \times \tilde{p}\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times p\mathfrak{M}_*(1-p) \times \tilde{p}\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p}$ are coordinates consistent with the bundle structure of $T^b(T_*(\frac{P_0 \times P_0}{G_0}))$.

- (iii) On $T_*P_0 \times T_*P_0$ one can take the coordinates $(\alpha_p, \beta_p, y_p, z_{pp_0}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0})$ which are the product of the coordinates $(\alpha_p, \beta_p, y_p, z_{pp_0}) \in p\mathfrak{M}_*(1-p) \times p\mathfrak{M}_*p \times (1-p)\mathfrak{M}p \times p\mathfrak{M}p_0$ and $(\tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0}) \in \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times \tilde{p}\mathfrak{M}_*\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p} \times \tilde{p}\mathfrak{M}p_0$ on T_*P_0 defined in (4.43). All components of (4.32) except of z_{pp_0} and $\tilde{z}_{\tilde{p}p_0}$ are G_0 -invariant. Hence one can consider

$$(\alpha_p, \beta_p, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \in p\mathfrak{M}_*(1-p) \times p\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times \tilde{p}\mathfrak{M}_*\tilde{p} \times (1-p)\mathfrak{M}p \times p\mathfrak{M}\tilde{p} \times (1-\tilde{p})\mathfrak{M}\tilde{p}, \quad (5.29)$$

where $z_{p\tilde{p}}$ is defined in (5.25), as a coordinates on $\frac{T_*P_0 \times T_*P_0}{G_0}$. The Poisson bracket of $f, g \in \mathcal{P}^\infty(\frac{T_*P_0 \times T_*P_0}{G_0})$ defined by the sub Poisson structure of $\frac{T_*P_0 \times T_*P_0}{G_0}$ written in the coordinates (5.29) assumes the following form

$$\begin{aligned} \{f, g\} = & \left\langle \frac{\partial g}{\partial y_p}, \frac{\partial f}{\partial \alpha_p} \right\rangle - \left\langle \frac{\partial f}{\partial y_p}, \frac{\partial g}{\partial \alpha_p} \right\rangle + \\ & + \left\langle \frac{\partial g}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial f}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial \tilde{y}_{\tilde{p}}}, \frac{\partial g}{\partial \tilde{\alpha}_{\tilde{p}}} \right\rangle + \left\langle \beta_p, \left[\frac{\partial g}{\partial \beta_p}, \frac{\partial f}{\partial \beta_p} \right] \right\rangle + \left\langle \tilde{\beta}_{\tilde{p}}, \left[\frac{\partial g}{\partial \tilde{\beta}_{\tilde{p}}}, \frac{\partial f}{\partial \tilde{\beta}_{\tilde{p}}} \right] \right\rangle + \\ & + \left\langle \frac{\partial g}{\partial z_{p\tilde{p}}}, \frac{\partial f}{\partial \beta_p} z_{p\tilde{p}} - z_{p\tilde{p}} \frac{\partial f}{\partial \tilde{\beta}_{\tilde{p}}} \right\rangle - \left\langle \frac{\partial f}{\partial z_{p\tilde{p}}}, \frac{\partial g}{\partial \beta_p} z_{p\tilde{p}} - z_{p\tilde{p}} \frac{\partial g}{\partial \tilde{\beta}_{\tilde{p}}} \right\rangle. \end{aligned} \quad (5.30)$$

Similarly as in (4.35-4.37) we have isomorphisms

$$T_{[(\varphi, \eta), (\psi, \xi)]} \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) \cong (\mathfrak{M}p_0)_* \times (1-p_0)\mathfrak{M}p_0 \times p\mathfrak{M}\tilde{p} \times (\mathfrak{M}p_0)_* \times (1-p_0)\mathfrak{M}p_0 \times \{[(\varphi, \eta), (\psi, \xi)]\} \quad (5.31)$$

$$T_{[(\varphi, \eta), (\psi, \xi)]}^* \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) \cong \mathfrak{M}p_0 \times p_0\mathfrak{M}^*(1-p_0) \times \tilde{p}\mathfrak{M}^*p \times \mathfrak{M}p_0 \times p_0\mathfrak{M}^*(1-p_0) \times \{[(\varphi, \eta), (\psi, \xi)]\} \quad (5.32)$$

$$T_{[(\varphi, \eta), (\psi, \xi)]}^b \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) \cong \mathfrak{M}p_0 \times p_0\mathfrak{M}_*(1-p_0) \times \tilde{p}\mathfrak{M}_*p \times \mathfrak{M}p_0 \times p_0\mathfrak{M}_*(1-p_0) \times \{[(\varphi, \eta), (\psi, \xi)]\} \quad (5.33)$$

where $[(\varphi, \eta), (\psi, \xi)] \in \frac{T_*P_0 \times T_*P_0}{G_0}$.

The coordinate expression for corresponding sub Poisson anchor $[\#_2] : T^b(\frac{T_*P_0 \times T_*P_0}{G_0}) \rightarrow T(\frac{T_*P_0 \times T_*P_0}{G_0})$ is

$$[\#_2](\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{\tilde{\beta}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{\tilde{\alpha}}_{\tilde{p}}, \overset{\circ}{\tilde{\beta}}_{\tilde{p}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}) = \quad (5.34)$$

$$= (-\overset{\circ}{y}_p, -ad_{\beta_p}^* (\beta_p) - z_{p\tilde{p}} \overset{\circ}{z}_{p\tilde{p}}, -\overset{\circ}{y}_{\tilde{p}}, -ad_{\tilde{\beta}_{\tilde{p}}}^* (\tilde{\beta}_{\tilde{p}}) + z_{p\tilde{p}} \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, z_{p\tilde{p}} - z_{p\tilde{p}} \overset{\circ}{\tilde{\beta}_{\tilde{p}}}, \overset{\circ}{\tilde{\alpha}_{\tilde{p}}}, \alpha_p, \beta_p, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}),$$

where

$$(\alpha_p, \beta_p, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \in p\mathfrak{M}_*(1-p) \times p\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \times \tilde{p}\mathfrak{M}_*\tilde{p} \times (1-p)\mathfrak{M}_p \times p\mathfrak{M}_*\tilde{p} \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \quad (5.35)$$

and

$$(\overset{\circ}{\alpha}_p, \overset{\circ}{\beta}_p, \overset{\circ}{\tilde{\alpha}_{\tilde{p}}}, \overset{\circ}{\tilde{\beta}_{\tilde{p}}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}_{\tilde{p}}}) \in (1-p)\mathfrak{M}_p \times p\mathfrak{M}_p \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \times \tilde{p}\mathfrak{M}_{\tilde{p}} \times p\mathfrak{M}_*(1-p) \times \tilde{p}\mathfrak{M}_*p \times \tilde{p}\mathfrak{M}_*(1-\tilde{p}) \quad (5.36)$$

are the coordinates along the fibres of $T^b(\frac{T_*P_0 \times T_*P_0}{G_0})$.

(iv) On $p_0\mathfrak{M}_*p_0 \times P_0 \times P_0$ we take the coordinates

$$(\chi, y_p, z_{pp_0}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0}) \in p_0\mathfrak{M}_*p_0 \times (1-p)\mathfrak{M}_p \times p\mathfrak{M}_p \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \times \tilde{p}\mathfrak{M}_{p_0}. \quad (5.37)$$

The Poisson bracket of $F, G \in \mathcal{P}^\infty(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0)$ in these coordinates has the form

$$\{F, G\} = - \left\langle \chi, \left[\frac{\partial F}{\partial \chi}, \frac{\partial G}{\partial \chi} \right] \right\rangle. \quad (5.38)$$

The coordinate expression of sub Poisson anchor in this case is

$$\begin{aligned} \#(\overset{\circ}{\mathcal{X}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{pp_0}, \overset{\circ}{\tilde{y}_{\tilde{p}}}, \overset{\circ}{z}_{\tilde{p}p_0}, \overset{\circ}{\mathcal{X}}, y_p, z_{pp_0}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0}) = \\ = (-ad_{\overset{\circ}{\mathcal{X}}}^* (\overset{\circ}{\mathcal{X}}), 0, 0, 0, 0, \overset{\circ}{\mathcal{X}}, y_p, z_{pp_0}, \tilde{y}_{\tilde{p}}, \tilde{z}_{\tilde{p}p_0}), \end{aligned} \quad (5.39)$$

where $(\overset{\circ}{\mathcal{X}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{pp_0}, \overset{\circ}{\tilde{y}_{\tilde{p}}}, \overset{\circ}{z}_{\tilde{p}p_0}) \in p_0\mathfrak{M}_*p_0 \times (1-p)\mathfrak{M}_p \times p\mathfrak{M}_p \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \times \tilde{p}\mathfrak{M}_{p_0}$ and

$$(\overset{\circ}{\mathcal{X}}, \overset{\circ}{y}_p, \overset{\circ}{z}_{pp_0}, \overset{\circ}{\tilde{y}_{\tilde{p}}}, \overset{\circ}{z}_{\tilde{p}p_0}) \in p_0\mathfrak{M}_{p_0} \times p\mathfrak{M}_*(1-p) \times p_0\mathfrak{M}_*p \times p\mathfrak{M}_*(1-\tilde{p}) \times \tilde{p}_0\mathfrak{M}_*p.$$

(v) As coordinates on $\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0}$ one can take

$$(\chi_p, y_p, \tilde{y}_{\tilde{p}}, z_{p\tilde{p}}) \in p_0\mathfrak{M}_*p_0 \times (1-p)\mathfrak{M}_p \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \times p\mathfrak{M}_{\tilde{p}} \quad (5.40)$$

where $z_{p\tilde{p}}$ is defined in (5.25) and

$$\chi_p := z_{pp_0} \chi z_{pp_0}^{-1}. \quad (5.41)$$

Hence for $F, G \in C^\infty(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0})$ the Poisson bracket (5.38) in the coordinates (5.40) has the form

$$\{F, G\}(\chi_p, y_p, \tilde{y}_{\tilde{p}}, z_{p\tilde{p}}) = - \left\langle \chi_p, \left[\frac{\partial F}{\partial \chi_p}(\chi_p, y_p, \tilde{y}_{\tilde{p}}, z_{p\tilde{p}}), \frac{\partial G}{\partial \chi_p}(\chi_p, y_p, \tilde{y}_{\tilde{p}}, z_{p\tilde{p}}) \right] \right\rangle. \quad (5.42)$$

We have isomorphisms

$$T_{[(x, \xi, \eta)]} \left(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0} \right) \cong p_0\mathfrak{M}_*p_0 \times (1-p)\mathfrak{M}_p \times (1-\tilde{p})\mathfrak{M}_{\tilde{p}} \times p\mathfrak{M}_{\tilde{p}} \times [(x, \xi, \eta)],$$

$$T_{[(x, \xi, \eta)]}^* \left(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0} \right) \cong p_0\mathfrak{M}_{p_0} \times p\mathfrak{M}^*(1-p) \times \tilde{p}\mathfrak{M}^*(1-\tilde{p}) \times \tilde{p}\mathfrak{M}^*p \times [(x, \xi, \eta)],$$

$$T_{[(x,\xi,\eta)]}^b \left(\frac{p_0 \mathfrak{M}_* p_0 \times P_0 \times P_0}{G_0} \right) \cong p_0 \mathfrak{M} p_0 \times p \mathfrak{M}_* (1-p) \times \tilde{p} \mathfrak{M}_* (1-\tilde{p}) \times \tilde{p} \mathfrak{M}_* p \times [(x, \xi, \eta)]. \quad (5.43)$$

In this case the sub Poisson anchor is given by

$$\#(\overset{\circ}{\mathcal{X}}_p, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}, \mathcal{X}_p, y_p, \tilde{z}_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) = (-ad_{\overset{\circ}{\mathcal{X}}_p}^* (\mathcal{X}_p), 0, 0, 0, \mathcal{X}_p, y_p, \tilde{z}_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \quad (5.44)$$

where $(\overset{\circ}{\mathcal{X}}_p, \overset{\circ}{y}_p, \overset{\circ}{z}_{p\tilde{p}}, \overset{\circ}{\tilde{y}}_{\tilde{p}}, \mathcal{X}_p, y_p, \tilde{z}_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \in p_0 \mathfrak{M} p_0 \times p \mathfrak{M}^* (1-p) \times \tilde{p} \mathfrak{M}^* p \times \tilde{p} \mathfrak{M}^* (1-\tilde{p}) \times p_0 \mathfrak{M}_* p_0 \times (1-p) \mathfrak{M} p \times p \mathfrak{M} \tilde{p} \times (1-\tilde{p}) \mathfrak{M} \tilde{p}$.

Let us mention here that the variables which are marked above by \circ concern those parts of coordinate systems which are taken along the fibres of the considered T^b - bundles.

The morphisms of \mathcal{VB} -groupoids, i.e. the horizontal arrows of (5.20) as well as their structural maps written in the coordinates listed above assume exceptionally simple forms. Namely for a_2^* and ι_2^* we have

$$\begin{aligned} (\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) &= a_2^*(\alpha_p, \beta_{\tilde{p}p}, \tilde{\alpha}_{\tilde{p}}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) = \\ &= (\alpha_p, z_{p\tilde{p}} \beta_{\tilde{p}p}, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, -\tilde{\beta}_{\tilde{p}p} z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \end{aligned} \quad (5.45)$$

and

$$(\chi_p, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) = \iota_2^*(\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) = (\beta_p + z_{p\tilde{p}} \tilde{\beta}_{\tilde{p}} z_{p\tilde{p}}^{-1}, y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}), \quad (5.46)$$

respectively.

The structural maps of the \mathcal{VB} -groupoid $\frac{T_* P_0 \times T_* P_0}{G_0} \rightrightarrows T_* P_0 / G_0$ which are obtained by the quotienting of the structural maps (5.14), written in the coordinates $(\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}})$, assume the following form

$$\tilde{s}_*(\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) = (-\tilde{\alpha}_{\tilde{p}}, -\tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) \quad (5.47)$$

$$\tilde{t}_*(\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) = (\alpha_p, \beta_p, y_p) \quad (5.48)$$

$$\tilde{\mathbf{I}}_*(\alpha_p, \beta_p, y_p) = (\alpha_p, \beta_p, y_p, p, -\alpha_p, -\beta_p, y_p) \quad (5.49)$$

$$\lambda_*(\alpha_p, \beta_p, y_p) = y_p \quad (5.50)$$

$$0_*(y_p) = (0, 0, y_p) \quad (5.51)$$

$$\tilde{\lambda}_*(\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, \tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) = (y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) \quad (5.52)$$

$$\tilde{0}_*(y_p, z_{p\tilde{p}}, \tilde{y}_{\tilde{p}}) = (0, 0, y_p, z_{p\tilde{p}}, 0, 0, \tilde{y}_{\tilde{p}}) \quad (5.53)$$

and the groupoid product is given by

$$\begin{aligned} (\alpha_p, \beta_p, y_p, z_{p\tilde{p}}, -\tilde{\alpha}_{\tilde{p}}, -\tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}) \cdot (\tilde{\alpha}_{\tilde{p}}, \tilde{\beta}_{\tilde{p}}, \tilde{y}_{\tilde{p}}, z_{\tilde{p}\tilde{p}}, \tilde{\tilde{\alpha}}_{\tilde{\tilde{p}}}, \tilde{\tilde{\beta}}_{\tilde{\tilde{p}}}, \tilde{\tilde{y}}_{\tilde{\tilde{p}}}) = \\ = (\alpha_p, \beta_p, y_p, z_{p\tilde{p}} z_{\tilde{p}\tilde{p}}, \tilde{\tilde{\alpha}}_{\tilde{\tilde{p}}}, \tilde{\tilde{\beta}}_{\tilde{\tilde{p}}}, \tilde{\tilde{y}}_{\tilde{\tilde{p}}}). \end{aligned}$$

Now let us consider the \mathcal{VB} -groupoid

$$\begin{array}{ccc} T(T_* P_0 \times T_* P_0) & \longrightarrow & T_* P_0 \times T_* P_0 \\ \begin{array}{c} \downarrow T\tilde{t}_* \\ \downarrow T\tilde{s}_* \end{array} & & \begin{array}{c} \downarrow \tilde{t}_* \\ \downarrow \tilde{s}_* \end{array} \\ T(T_* P_0) & \longrightarrow & T_* P_0 \end{array}, \quad (5.54)$$

which is the tangent prolongation of the groupoid $T_*P_0 \times T_*P_0 \rightrightarrows T_*P_0$, which is a subgroupoid of the groupoid $T^*P_0 \times T^*P_0 \rightrightarrows T^*P_0$. So, its structure is defined in (5.14) and (5.15). Taking into account isomorphisms

$$T(T_*P_0) \cong p_0\mathfrak{M}_* \times \mathfrak{M}_{p_0} \times p_0\mathfrak{M}_* \times P_0 \quad (5.55)$$

and

$$T(T_*P_0 \times T_*P_0) \cong (\mathfrak{M}_{p_0})_* \times \mathfrak{M}_{p_0} \times (\mathfrak{M}_{p_0})_* \times \mathfrak{M}_{p_0} \times (\mathfrak{M}_{p_0})_* \times P_0 \times (\mathfrak{M}_{p_0})_* \times P_0 \quad (5.56)$$

we find that the structural maps of (5.54) are:

$$\begin{aligned} T\tilde{\mathbf{s}}_*(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi) &= (-\dot{\psi}, \dot{\xi}, -\psi, \xi) \\ T\tilde{\mathbf{t}}_*(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi) &= (\dot{\varphi}, \dot{\eta}, \varphi, \eta) \\ T\tilde{\mathbf{l}}_*(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi) &= (-\dot{\psi}, \dot{\xi}, -\dot{\varphi}, \dot{\eta}, -\psi, \xi, -\varphi, \eta) \\ T\tilde{\mathbf{1}}_*(\dot{\varphi}, \dot{\eta}, \varphi, \eta) &= (\dot{\varphi}, \dot{\eta}, -\dot{\varphi}, \dot{\eta}, \varphi, \eta, -\varphi, \eta) \\ \tilde{\lambda}(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi) &= (\varphi, \eta, \psi, \xi) \\ \lambda(\dot{\varphi}, \dot{\eta}, \varphi, \eta) &= (\varphi, \eta) \\ \tilde{0}(\varphi, \eta, \psi, \xi) &= (0, 0, 0, 0, \varphi, \eta, \psi, \xi) \\ 0(\varphi, \eta) &= (0, 0, \varphi, \eta) \end{aligned} \quad (5.57)$$

and its groupoid product is given by

$$(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi)(-\dot{\psi}, \dot{\xi}, \dot{\sigma}, \dot{\lambda}, -\psi, \xi, \sigma, \lambda) = (\dot{\varphi}, \dot{\eta}, \dot{\sigma}, \dot{\lambda}, \varphi, \eta, \sigma, \lambda). \quad (5.58)$$

One easy sees that the core of (5.54) is isomorphic with $T(T_*P_0)$. So, the dual \mathcal{VB} -groupoid of (5.54) is

$$\begin{array}{ccc} T^*(T_*P_0 \times T_*P_0) & \longrightarrow & T_*P_0 \times T_*P_0 \\ \downarrow T^*\tilde{\mathbf{t}}_* & & \downarrow \tilde{\mathbf{t}}_* \\ T^*(T_*P_0) & \longrightarrow & T_*P_0 \end{array} \quad \begin{array}{ccc} & & \downarrow \tilde{\mathbf{s}}_* \\ & & T_*P_0 \end{array} \quad (5.59)$$

Using isomorphisms:

$$T^*(T_*P_0) \cong \mathfrak{M}_{p_0} \times (\mathfrak{M}_{p_0})^* \times (\mathfrak{M}_{p_0})_* \times P_0 \quad (5.60)$$

and

$$T^*(T_*P_0 \times T_*P_0) \cong (\mathfrak{M}_{p_0}) \times (\mathfrak{M}_{p_0})^* \times \mathfrak{M}_{p_0} \times (\mathfrak{M}_{p_0})^* \times (\mathfrak{M}_{p_0})_* \times P_0 \times (\mathfrak{M}_{p_0})_* \times P_0 \quad (5.61)$$

we write the structural maps of (5.59) as follows:

$$\begin{aligned} T^*\tilde{\mathbf{s}}_*(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= (\overset{\circ}{\psi}, -\overset{\circ}{\xi}, -\psi, \xi) \\ T^*\tilde{\mathbf{t}}_*(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= (\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) \\ T^*\tilde{\mathbf{l}}_*(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= (\overset{\circ}{\psi}, -\overset{\circ}{\xi}, \overset{\circ}{\varphi}, -\overset{\circ}{\eta}, -\psi, \xi, -\varphi, \eta) \\ T^*\tilde{\mathbf{1}}_*(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) &= (\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\eta}, \varphi, \eta, -\varphi, \eta) \\ \tilde{\lambda}(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= (\varphi, \eta, \psi, \xi) \\ \lambda(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) &= (\varphi, \eta) \\ \tilde{0}(\varphi, \eta, \psi, \xi) &= (0, 0, 0, 0, \varphi, \eta, \psi, \xi) \\ 0(\varphi, \eta) &= (0, 0, \varphi, \eta) \end{aligned} \quad (5.62)$$

and the groupoid product is given by

$$(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)(\overset{\circ}{\psi}, -\overset{\circ}{\xi}, \overset{\circ}{\sigma}, \overset{\circ}{\lambda}, -\psi, \xi, \sigma, \lambda) = (\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\sigma}, \overset{\circ}{\lambda}, \varphi, \eta, \sigma, \lambda). \quad (5.63)$$

The quasi Banach subbundles $T^b(T_*P_0) \subset T^*(T_*P_0)$ and $T^b(T_*P_0 \times T_*P_0) \subset T^*(T_*P_0 \times T_*P_0)$ are isomorphic to

$$T^b(T_*P_0) \cong \mathfrak{M}_{p_0} \times p_0\mathfrak{M}_* \times p_0\mathfrak{M}_* \times P_0 \quad (5.64)$$

and

$$T^b(T_*P_0 \times T_*P_0) \cong (\mathfrak{M}_{p_0}) \times (\mathfrak{M}_{p_0})_* \times \mathfrak{M}_{p_0} \times (\mathfrak{M}_{p_0})_* \times (\mathfrak{M}_{p_0})_* \times P_0 \times (\mathfrak{M}_{p_0})_* \times P_0, \quad (5.65)$$

respectively. Using the isomorphisms (5.55), (5.56), (5.64) and (5.65) we write the sub Poisson maps $\#_1 : T^b(T_*P_0) \rightarrow T(T_*P_0)$ and $\#_2 : T^b(T_*P_0 \times T_*P_0) \rightarrow T(T_*P_0 \times T_*P_0)$, which are defined by the brackets (4.71) and (5.22), as follows

$$\#_1(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) = (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, \varphi, \eta). \quad (5.66)$$

and

$$\#_2(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) = (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\xi}, \overset{\circ}{\psi}, \varphi, \eta, \psi, \xi). \quad (5.67)$$

We note here that the \mathcal{VB} -subgroupoid

$$\begin{array}{ccc} T^b(T_*P_0 \times T_*P_0) & \longrightarrow & T^*(T_*P_0 \times T_*P_0) \\ \downarrow \downarrow & & \downarrow \downarrow \\ T^b(T_*P_0) & \longrightarrow & T^*(T_*P_0) \end{array} \quad (5.68)$$

of the \mathcal{VB} -groupoid (5.59) will be crucial for the following considerations. Also the momentum maps $J_{1b} : T^b(T_*P_0) \rightarrow p_0\mathfrak{M}_*p_0$ and $J_{2b} : T^b(T_*P_0 \times T_*P_0) \rightarrow p_0\mathfrak{M}_*p_0$ which are given by

$$J_{1b}(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) = \overset{\circ}{\eta} \eta - \varphi \overset{\circ}{\varphi} \quad (5.69)$$

and

$$J_{2b}(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) = \overset{\circ}{\eta} \eta - \varphi \overset{\circ}{\varphi} + \overset{\circ}{\xi} \xi - \psi \overset{\circ}{\psi} \quad (5.70)$$

will be important in subseque.

One has a sequence of quasi Banach vector subbundles

$$\begin{array}{ccccccc} (T^*\tilde{\mathfrak{t}}_*)^{-1}(J_{1b}^{-1}(0)) \cap (T^*\tilde{\mathfrak{s}}_*)^{-1}(J_{1b}^{-1}(0)) & \longrightarrow & J_{2b}^{-1}(0) & \longrightarrow & T^b(T_*P_0 \times T_*P_0) \\ \downarrow & & \downarrow & & \downarrow \\ T_*P_0 \times T_*P_0 & \longrightarrow & T_*P_0 \times T_*P_0 & \longrightarrow & T_*P_0 \times T_*P_0 \end{array} \quad (5.71)$$

of the vector bundle $T^b(T_*P_0 \times T_*P_0) \rightarrow T_*P_0 \times T_*P_0$.

Let us define the bundle $\mathfrak{J} \rightarrow J_2^{-1}(0)$ as the restriction of the first subbundle in (5.71) to the submanifold $J_2^{-1}(0) \hookrightarrow T_*P_0 \times T_*P_0$.

Lemma 5.7. *One has the following sequence of \mathcal{VB} -groupoids morphisms*

$$\begin{array}{ccccccccc}
\mathfrak{J} & \xrightarrow{\iota_2} & J_{2b}^{-1}(0) & \xrightarrow{\tilde{\iota}_2} & T^\flat(T_*P_0 \times T_*P_0) & \xrightarrow{\#_2} & T(T_*P_0 \times T_*P_0) & \xrightarrow{Q_2} & \frac{T(T_*P_0 \times T_*P_0)}{T_e G_0} \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
J_{1b}^{-1}(0) & \xrightarrow{\iota_1} & T^\flat(T_*P_0) & \xrightarrow{id} & T^\flat(T_*P_0) & \xrightarrow{\#_1} & T(T_*P_0) & \xrightarrow{Q_1} & \frac{T(T_*P_0)}{T_e G_0},
\end{array} \tag{5.72}$$

where $\#_1$ and $\#_2$ are as in (5.66) and (5.67). The maps Q_1 and Q_2 are the respective quotient maps, see (5.81) and (5.82). The \mathcal{VB} -groupoid $\mathfrak{J} \rightrightarrows J_{1b}^{-1}(0)$ has $J_{2b}^{-1}(0) \rightrightarrows T_*P_0$ as its side groupoid. The side groupoid of others \mathcal{VB} -groupoids in (5.72) is $T_*P_0 \times T_*P_0 \rightrightarrows T_*P_0$.

Proof. The equalities

$$\begin{aligned}
(T\tilde{\mathbf{t}}_* \circ \#_2)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= T\tilde{\mathbf{t}}_*(-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\xi}, \overset{\circ}{\psi}, \varphi, \eta, \psi, \xi) = \\
&= (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, \varphi, \eta) = \#_1(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) = (\#_1 \circ T^*\tilde{\mathbf{t}}_*)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)
\end{aligned} \tag{5.73}$$

$$\begin{aligned}
(T\tilde{\mathbf{s}}_* \circ \#_2)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= T\tilde{\mathbf{s}}_*(-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\xi}, \overset{\circ}{\psi}, \varphi, \eta, \psi, \xi) = \\
&= (\overset{\circ}{\xi}, \overset{\circ}{\psi}, -\psi, \xi) = \#_1(\overset{\circ}{\psi}, -\overset{\circ}{\xi}, -\psi, \xi) = (\#_1 \circ T^*\tilde{\mathbf{s}}_*)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)
\end{aligned} \tag{5.74}$$

$$\begin{aligned}
(\#_2 \circ T^*\tilde{\iota}_*)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) &= \#_2(\overset{\circ}{\psi}, -\overset{\circ}{\xi}, \overset{\circ}{\varphi}, -\overset{\circ}{\eta}, -\psi, \xi, -\varphi, \eta) = (\overset{\circ}{\xi}, \overset{\circ}{\psi}, \overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\phi, \xi, -\varphi, \eta) = \\
&= T\tilde{\iota}_*(-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\xi}, \overset{\circ}{\psi}, \varphi, \eta, \psi, \xi) = (T\tilde{\iota}_* \circ \#_2)(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)
\end{aligned} \tag{5.75}$$

$$\begin{aligned}
(\#_2(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)) \cdot (\#_2(\overset{\circ}{\psi}, -\overset{\circ}{\xi}, \overset{\circ}{\sigma}, \overset{\circ}{\lambda}, -\psi, \xi, \sigma, \lambda)) &= \\
&= (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\xi}, \overset{\circ}{\psi}, \varphi, \eta, \psi, \xi)(\overset{\circ}{\xi}, \overset{\circ}{\psi}, -\overset{\circ}{\lambda}, \overset{\circ}{\sigma}, -\psi, \xi, \sigma, \lambda) = \\
&= (-\overset{\circ}{\eta}, \overset{\circ}{\varphi}, -\overset{\circ}{\lambda}, \overset{\circ}{\sigma}, \varphi, \eta, \sigma, \lambda) = \#_2(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\sigma}, \overset{\circ}{\lambda}, \varphi, \eta, \sigma, \lambda) = \\
&= \#_2 \left((\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi)(\overset{\circ}{\psi}, -\overset{\circ}{\xi}, \overset{\circ}{\sigma}, \overset{\circ}{\lambda}, -\psi, \xi, \sigma, \lambda) \right)
\end{aligned} \tag{5.76}$$

show that $\#_1$ and $\#_2$ define the groupoids morphism

$$\begin{array}{ccc}
T^\flat(T_*P_0 \times T_*P_0) & \xrightarrow{\#_2} & T(T_*P_0 \times T_*P_0) \\
\begin{array}{c} \mathbf{t}^* \parallel \mathbf{s}^* \\ \downarrow \downarrow \end{array} & & \begin{array}{c} T\tilde{\mathbf{t}}_* \parallel T\tilde{\mathbf{s}}_* \\ \downarrow \downarrow \end{array} \\
T^\flat(T_*P_0) & \xrightarrow{\#_1} & T(T_*P_0)
\end{array} \tag{5.77}$$

In order to see that (ι_1, ι_2) and $(id, \tilde{\iota}_2)$ define the groupoids morphisms we note that the coordinate description of the considered manifolds is the following

$$J_{1b}^{-1}(0) = \{(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \varphi, \eta) : \overset{\circ}{\eta} \eta - \varphi \overset{\circ}{\varphi} = 0\}. \tag{5.78}$$

$$J_{2b}^{-1}(0) = \{(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) : \overset{\circ}{\eta} \eta - \varphi \overset{\circ}{\varphi} + \overset{\circ}{\xi} \xi - \psi \overset{\circ}{\psi} = 0\} \quad (5.79)$$

and

$$\mathfrak{J} = \{(\overset{\circ}{\varphi}, \overset{\circ}{\eta}, \overset{\circ}{\psi}, \overset{\circ}{\xi}, \varphi, \eta, \psi, \xi) : \overset{\circ}{\eta} \eta - \varphi \overset{\circ}{\varphi} = 0, \quad \overset{\circ}{\xi} \xi - \psi \overset{\circ}{\psi} = 0, \quad \varphi \eta + \psi \xi = 0\}. \quad (5.80)$$

Next we note that the conditions mentioned in (5.78-5.80) are invariant with respect to the groupoids operations. The quotient maps Q_1 and Q_2 are defined by

$$Q_1(\dot{\varphi}, \dot{\eta}, \varphi, \eta) := [(\dot{\varphi}, \dot{\eta}, \varphi, \eta)] = \{(\dot{\varphi} - x\varphi, \dot{\eta} + \eta x, \varphi, \eta) : x \in p_0 \mathfrak{M} p_0\} \quad (5.81)$$

$$Q_2(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi) := [(\dot{\varphi}, \dot{\eta}, \dot{\psi}, \dot{\xi}, \varphi, \eta, \psi, \xi)] = \{(\dot{\varphi} - x\varphi, \dot{\eta} + \eta x, \dot{\psi} - x\psi, \dot{\xi} + \xi x, \varphi, \eta, \psi, \xi) : x \in p_0 \mathfrak{M} p_0\}. \quad (5.82)$$

Using (5.81) and (5.82) we can show by the directly calculation that (Q_1, Q_2) also defines a groupoid morphism. □

Now we consider the \mathcal{VB} -groupoid

$$\begin{array}{ccc} T(p_0 \mathfrak{M}_* p_0 \times P_0 \times P_0) & \longrightarrow & p_0 \mathfrak{M}_* p_0 \times P_0 \times P_0 \\ \downarrow \downarrow & & \downarrow \downarrow \\ T(\{0\}^* \times P_0) & \longrightarrow & \{0\}^* \times P_0 \end{array} \quad (5.83)$$

tangent to the \mathcal{VB} - groupoid (5.16). The structural maps for (5.83) are the following:

$$\begin{aligned} T\tilde{\mathbf{s}}_*(\dot{\chi}, \chi, v, \eta, w, \xi) &:= (0, w, \xi), \\ T\tilde{\mathbf{t}}_*(\dot{\chi}, \chi, v, \eta, w, \xi) &:= (0, v, \eta), \\ T\tilde{\mathbf{l}}_*(\chi, v, \eta) &:= (0, 0, v, \eta, v, \eta), \\ T\tilde{\lambda}_*(\dot{\chi}, \chi, v, \eta, w, \xi) &= (\chi, \eta, \xi), \\ \tilde{0}_*(\chi, \eta, \xi) &= (0, \chi, 0, \eta, 0, \xi) \\ \lambda_*(\chi, v, \eta) &= (0, \eta), \\ 0_*(0, \eta) &= (0, 0, \eta) \end{aligned} \quad (5.84)$$

and the inverse map and the groupoid product are

$$\begin{aligned} T\tilde{\iota}_*(\dot{\chi}, \chi, v, \eta, w, \xi) &:= (-\dot{\chi}, -\chi, w, \xi, v, \eta), \\ (\dot{\chi}, \chi, v, \eta, w, \xi)(\mathcal{Y}, \mathcal{Z}, w, \xi, z, \zeta) &:= (\dot{\chi} + \mathcal{Y}, \chi + \mathcal{Z}, v, \eta, z, \zeta). \end{aligned} \quad (5.85)$$

For the \mathcal{VB} -groupoid

$$\begin{array}{ccc} T^*(p_0 \mathfrak{M}_* p_0 \times P_0 \times P_0) & \longrightarrow & p_0 \mathfrak{M}_* p_0 \times P_0 \times P_0 \\ \downarrow \downarrow & & \downarrow \downarrow \\ p_0 \mathfrak{M}_* p_0 \times T^* P_0 & \longrightarrow & \{0\} \times P_0 \end{array} \quad (5.86)$$

being the dualization of (5.83) the structural maps and the groupoid product are the following:

$$\begin{aligned}
T^*\tilde{\mathbf{s}}_*(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) &:= (\overset{\circ}{\chi}, -\psi, \xi), \\
T\tilde{\mathbf{t}}_*(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) &:= (\overset{\circ}{\chi}, \varphi, \eta), \\
T\tilde{\mathbf{l}}_*(\overset{\circ}{\chi}, \chi, \varphi, \eta) &:= (\overset{\circ}{\chi}, 0, -\varphi, \eta, \varphi, \eta), \\
\tilde{\lambda}_*(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) &= (\chi, \eta, \xi), \\
\tilde{0}_*(\chi, \eta, \xi) &= (0, \chi, 0, \eta, 0, \xi) \\
\lambda_*(\chi, \varphi, \eta) &= (0, \eta), \\
0_*(0, \eta) &= (0, 0, \eta).
\end{aligned} \tag{5.87}$$

The inverse map and groupoid product for (5.86) are given by

$$\begin{aligned}
T\tilde{\iota}_*(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) &:= (\overset{\circ}{\chi}, -\chi, -\psi, \xi, \varphi, \eta), \\
(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi)(\overset{\circ}{\chi}, \mathcal{Y}, -\psi, \xi, \theta, \zeta) &:= (\overset{\circ}{\chi}, \chi + \mathcal{Y}, \varphi, \eta, \theta, \zeta).
\end{aligned} \tag{5.88}$$

Lemma 5.8. *One has the following sequence of \mathcal{VB} -groupoids morphisms*

$$\begin{array}{ccccccc}
J_b^{-1}(0) & \xrightarrow{\iota_2} & T^b(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0) & \xrightarrow{\tilde{\#}_2} & T(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0) & \xrightarrow{\tilde{Q}_2} & \frac{T(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0)}{T_e G_0} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
p_0\mathfrak{M}p_0 \times T^b P_0 & \xrightarrow{id} & p_0\mathfrak{M}p_0 \times T^b P_0 & \xrightarrow{\tilde{a}_*} & \{0\} \times T P_0 & \xrightarrow{\tilde{Q}_1} & \{0\} \times \frac{T P_0}{T_e G_0},
\end{array} \tag{5.89}$$

where $J_b : T^b(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0) \rightarrow p_0\mathfrak{M}_*p_0$ is defined by

$$J_b(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) := ad_{\overset{\circ}{\chi}}^*(\chi) + \varphi\eta + \psi\xi, \tag{5.90}$$

for $T^b(p_0\mathfrak{M}_*p_0 \times P_0 \times P_0) \cong T^*(p_0\mathfrak{M}_*p_0) \times T_*(P_0 \times P_0)$. The anchor maps $\tilde{\#}_2$ and \tilde{a}_* are defined by

$$\tilde{\#}_2(\overset{\circ}{\chi}, \chi, \varphi, \eta, \psi, \xi) := (-ad_{\overset{\circ}{\chi}}^*(\chi), \chi, 0, \eta, 0, \xi) \tag{5.91}$$

$$\tilde{a}_*(\overset{\circ}{\chi}, \varphi, \eta) := (0, \eta), \tag{5.92}$$

respectively. The bundle morphisms \tilde{Q}_2 and \tilde{Q}_1 in (5.89) are the projections on the respective quotient bundles given by

$$\tilde{Q}_1(0, v, \eta) = [(0, v, \eta)] = \{(0, v + \eta x, \eta); \quad x \in p_0\mathfrak{M}p_0\} \tag{5.93}$$

$$\tilde{Q}_2(\overset{\circ}{\chi}, \chi, v, \eta, w, \xi) = [(\overset{\circ}{\chi}, \chi, v, \eta, w, \xi)] = \{(\overset{\circ}{\chi} + ad_{\overset{\circ}{\chi}}^*(\chi), \chi, v + \eta x, \eta, w + \xi x, \xi); \quad x \in p_0\mathfrak{M}p_0\}. \tag{5.94}$$

Proof. By the direct verification. \square

The following theorem summarizes important facts concerning the fibre-wise linear sub Poisson structures related to the gauge groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$. Recall that this gauge groupoid is isomorphic to Banach-Lie groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$.

Theorem 5.9. *All Banach-Lie groupoids in the front of the spatial diagram (5.20) are sub Poisson groupoids and the corresponding horizontal arrows of (5.20) define sub Poisson morphisms between of them.*

Proof. All horizontal maps in (5.72) are G_0 -equivariant groupoid morphisms. So, quotienting (5.72) by G_0 we obtain a sequence of morphisms of the quotient groupoids. Let us describe these groupoids.

We have the following bundle isomorphisms:

$$J_{2b}^{-1}(0)/G_0 \cong T^b \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right), \quad (5.95)$$

$$\frac{T(T_*P_0 \times T_*P_0)}{T_e G_0}/G_0 \cong T \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right), \quad (5.96)$$

$$\frac{T(T_*P_0)}{T_e G_0}/G_0 \cong T \left(\frac{T_*P_0}{G_0} \right). \quad (5.97)$$

Thus, the quotienting of $J_{2b}^{-1}(0) \rightrightarrows T^b(T_*P_0)$ and $\frac{T(T_*P_0 \times T_*P_0)}{T_e G_0} \rightrightarrows \frac{T(T_*P_0)}{T_e G_0}$ by G_0 leads to the groupoid morphism

$$\begin{array}{ccc} T^b \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) & \xrightarrow{[\#_2]} & T \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) \\ \parallel & & \parallel \\ \frac{T^b(T_*P_0)}{G_0} & \xrightarrow{[\#_1]} & T \left(\frac{T_*P_0}{G_0} \right). \end{array} \quad (5.98)$$

where the sub Poisson anchors $[\#_1]$ and $[\#_2]$ in (5.98) are defined as quotients of morphisms $Q_1 \circ \#_1 \circ id$ and $Q_2 \circ \#_2 \circ \tilde{i}_2$ by G_0 , respectively. From (5.98) we conclude that $\frac{T_*P_0 \times T_*P_0}{G_0} \rightrightarrows T_*P_0/G_0$ is a sub Poisson \mathcal{VB} -groupoid. Let us note here that the bundle $J_{2b}^{-1}(0) \rightarrow T_*P_0 \times T_*P_0$ can be considered as the bundle dual to $T \left(\frac{T_*P_0 \times T_*P_0}{G_0} \right) \rightarrow T_*P_0 \times T_*P_0$.

The quotient groupoid of the first \mathcal{VB} -groupoid in (5.72) is isomorphic to the \mathcal{VB} -groupoid

$$\begin{array}{ccc} T^b \left(T_* \left(\frac{P_0 \times P_0}{G_0} \right) \right) & \longrightarrow & T_* \left(\frac{P_0 \times P_0}{G_0} \right) \\ \parallel & & \parallel \\ T^b \left(\frac{T_*P_0}{G_0} \right) & \longrightarrow & \frac{T_*P_0}{G_0}. \end{array} \quad (5.99)$$

which has $T_* \left(\frac{P_0 \times P_0}{G_0} \right) \rightrightarrows T_*P_0/G_0$ as its side groupoid. Thus after quotienting (5.72) by G_0 we obtain the \mathcal{VB} -groupoid morphism

$$\begin{array}{ccc} T^b \left(T_* \left(\frac{P_0 \times P_0}{G_0} \right) \right) & \xrightarrow{\tilde{\#}_2} & T \left(T_* \left(\frac{P_0 \times P_0}{G_0} \right) \right) \\ \parallel & & \parallel \\ T^b \left(\frac{T_*P_0}{G_0} \right) & \xrightarrow{\tilde{\#}_1} & T \left(\frac{T_*P_0}{G_0} \right). \end{array} \quad (5.100)$$

where $\tilde{\#}_1$ and $\tilde{\#}_2$ are defined as the quotients of $Q_1 \circ \#_1 \circ id \circ \iota_1$ and $Q_2 \circ \#_2 \circ \tilde{\iota}_2 \circ \iota_2$ by G_0 , respectively. Note here that $\tilde{\#}_1$ and $\tilde{\#}_2$ expressed in G_0 -invariant coordinates assume the form presented in (4.75) and in (5.28). We also recall that $\#_2 : T^b\left(\frac{T_*P_0 \times T_*P_0}{G_0}\right) \rightarrow T\left(\frac{T_*P_0 \times T_*P_0}{G_0}\right)$ maps elements of $T^b\left(\frac{T_*P_0 \times T_*P_0}{G_0}\right)$ onto vectors tangent to the symplectic leaves of $\frac{T_*P_0 \times T_*P_0}{G_0}$, so, in the particular case onto vectors tangent to $T_*\left(\frac{P_0 \times P_0}{G_0}\right) \cong J_{2b}^{-1}(0)/G_0 \subset \frac{T_*P_0 \times T_*P_0}{G_0}$. Therefore, it follows from (5.100) that $T_*\left(\frac{P_0 \times P_0}{G_0}\right) \rightrightarrows T_*P_0/G_0$ is a weak symplectic groupoid.

Taking the quotient of (5.89) by G_0 we obtain the \mathcal{VB} -groupoids morphisms

$$\begin{array}{ccc} T^b\left(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0}\right) & \xrightarrow{[\#]} & T\left(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0}\right) \\ \downarrow \downarrow & & \downarrow \downarrow \\ \frac{p_0\mathfrak{M}_*p_0 \times T^bP_0}{G_0} & \xrightarrow{[\tilde{a}_*]} & T(P_0/G_0) \end{array} \quad (5.101)$$

where $[\#] := [\tilde{Q}_2 \circ \# \circ \iota]$ and $[\tilde{a}_*] := [\tilde{Q}_1 \circ \tilde{a}_* \circ id]$ are the projectivizations of the respective maps from diagram (5.89). Note that in order to obtain (5.101) we have used the bundle isomorphism

$$J_b^{-1}(0)/G_0 \cong T^b\left(\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0}\right). \quad (5.102)$$

From (5.101) we find that $\frac{p_0\mathfrak{M}_*p_0 \times P_0 \times P_0}{G_0} \rightrightarrows \{0\} \times P_0/G_0$ is a Poisson groupoid. Ending we note that T^b -subbundles of T^* -bundles were defined in (5.43).

One can check by the straightforward verification that the horizontal arrows in (5.20) define the \mathcal{VB} -groupoids morphism. Applying Theorem 4.5 to the case of the principal bundle $P_0 \times P_0 \rightarrow \frac{P_0 \times P_0}{G_0}$ we find that they are also the Poisson morphisms. \square

6 Concluding remarks

In this section we will present cursory review of those questions which were not touched on in the paper but are crucial for the theory investigated here. We begin describing interrelation between the Banach Lie algebroid structure of $\mathcal{AG}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ and the linear sub Poisson structure on $\mathcal{A}_*\mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{L}(\mathfrak{M})$ which is not so obvious in a not-reflexive Banach case.

6.1 Algebroid structure of TP_0/G_0 and the linear sub Poisson structure on T_*P_0/G_0 .

In the previous sections we investigated the Atiyah sequence (3.8), the predual Atiyah sequence (4.2) as well as the short exact sequences of \mathcal{VB} -groupoids (5.2) and (5.20). All of these sequences are defined in a canonical way by the structure of W^* -algebra \mathfrak{M} . The predual Atiyah sequence (3.8) is an ingredient of (5.20) which after the dualization gives the short exact sequence of \mathcal{VB} -groupoids (5.2). However, except of the case when \mathfrak{M} has the finite dimention, the dualizations of (5.2) and (3.8) does not give back (5.20) and (4.2), i.e. the dualization procedure is not reversive in general. Let us discuss this question closely comparing the algebroid structure of TP_0/G_0 with the sub Poisson structure on T_*P_0/G_0 .

The section $\mathcal{V} \in \Gamma_{G_0}^\infty TP_0 \cong \Gamma^\infty(TP_0/G_0)$ and the G_0 -invariant function $\rho \in C_{G_0}^\infty(P_0) \cong C^\infty(P_0/G_0)$ define a function $f_{\mathcal{V},\rho} \in C^\infty(T_*P_0/G_0)$ on T_*P_0/G_0 by

$$f_{\mathcal{V},\rho}(\varphi, \eta) := \langle \varphi, \mathcal{V}(\eta) \rangle + \rho(\eta). \quad (6.1)$$

The bracket (4.23) taken on $f_{\mathcal{V}_1,\rho_1}, f_{\mathcal{V}_2,\rho_2} \in C^\infty(T_*P_0/G_0)$ fulfilled the equality

$$\{f_{\mathcal{V}_1,\rho_1}, f_{\mathcal{V}_2,\rho_2}\} = f_{[\mathcal{V}_1,\mathcal{V}_2], \mathcal{V}_1(\rho_1) - \mathcal{V}_2(\rho_2)} \quad (6.2)$$

where $[\mathcal{V}_1, \mathcal{V}_2]$ is the Lie algebroid bracket given by (3.41). From (6.2) we conclude that the space $\mathcal{L}_{G_0}^\infty(T_*P_0)$ of functions $f_{\mathcal{V},\rho}$ is a Lie algebra.

Now let us observe that the derivation $\{f_{\mathcal{V},\rho}, \cdot\}$ is a section of $T^{**}(T_*P_0)$ in general. It will be a vector field $\{f_{\mathcal{V},\rho}, \cdot\} \in \Gamma^\infty T(T_*P_0) \subsetneq \Gamma^\infty T^{**}(T_*P_0)$ if and only if $f_{\mathcal{V},\rho} \in \mathcal{P}_{G_0}^\infty(T_*P_0) \cong \mathcal{P}^\infty(T_*P_0/G_0)$.

Remark 6.1. (i) The function $f_{\mathcal{V},\rho}$ belongs to $\mathcal{P}_{G_0}^\infty(T_*P_0)$ iff

$$\varphi \circ \frac{\partial \mathcal{V}}{\partial \eta}(\eta) \in p_0 \mathfrak{M}_* \quad \text{and} \quad \frac{\partial \rho}{\partial \eta}(\eta) \in p_0 \mathfrak{M}_* \quad (6.3)$$

for any $\varphi \in p_0 \mathfrak{M}_*$ and $\eta \in P_0$.

(ii) The Lie subalgebra $\mathcal{A}_{G_0}^\infty(T_*P_0) \subset \mathcal{L}_{G_0}^\infty(T_*P_0)$, consisting functions $f_{\mathcal{V}} := f_{\mathcal{V},\rho}$ such that $\rho = 0$, is isomorphic to the Lie algebra $\Gamma_{G_0}^\infty TP_0$ of the algebroid of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$.

The first condition in (6.3) means that the Banach subspace $p_0 \mathfrak{M}_* \subset (\mathfrak{M}p_0)^*$ is invariant with respect to the bounded operator $\frac{\partial \mathcal{V}}{\partial \eta}(\eta)^* \in L^\infty((\mathfrak{M}p_0)^*)$ dual to $\frac{\partial \mathcal{V}}{\partial \eta}(\eta) \in L^\infty(\mathfrak{M}p_0)$ what means that the operator $\frac{\partial \rho}{\partial \eta}(\eta)$ is continuous with respect to $\sigma(\mathfrak{M}p_0, p_0 \mathfrak{M}_*)$ -topology of $\mathfrak{M}p_0$.

The proposition presented below summaries some important properties of the Lie subalgebra $\mathcal{L}_{G_0}^\infty(T_*P_0) \cap \mathcal{P}_{G_0}^\infty(T_*P_0)$.

Proposition 6.2. (i) The Poisson algebra $\mathcal{P}_{G_0}^\infty(T_*P_0)$ is generated by functions from the Lie subalgebra $\mathcal{L}_{G_0}^\infty(T_*P_0) \cap \mathcal{P}_{G_0}^\infty(T_*P_0)$.

(ii) If $f \in \mathcal{L}_{G_0}^\infty(T_*P_0) \cap \mathcal{P}_{G_0}^\infty(T_*P_0)$ then the cotangent lift $L_t^* : T^*P_0 \rightarrow T^*P_0$ of the left translation flow $L_t : P_0 \rightarrow P_0$ tangent to $\mathcal{V} \in \Gamma_{G_0}^\infty(TP_0)$ preserves the precotangent bundle $T_*P_0 \subset T^*P_0$. The vector field $\{f_{\mathcal{V},\rho}, \cdot\} \in \Gamma^\infty T(T^*P_0)$ is tangent to L_t^* .

Taking T^*P_0/G_0 instead of T_*P_0/G_0 and applying the bracket (4.23) to the functions (6.1) now defined on T^*P_0 we find that the space of such functions $\mathcal{L}_{G_0}^\infty(T^*P_0)$ is a Lie algebra. In this case the derivation

$$\{f_{\mathcal{V}}, \cdot\} = \mathcal{V}(\eta) \frac{\partial}{\partial \eta} - \left\langle \varphi, \frac{\partial \mathcal{V}}{\partial \eta}(\eta) \right\rangle \frac{\partial}{\partial \rho} \quad (6.4)$$

is a linear vector field on T^*P_0 which, however, after restriction to $T_*P_0 \subset T^*P_0$ will be a section of $T^{**}(T_*P_0)$ in general. The vector field $\{f_{\mathcal{V}}, \cdot\} \in \Gamma^\infty T(T^*P_0)$ is tangent to L_t^* .

Summing up we conclude that in the case considered here the correspondence between the sections of the algebroid TP_0/G_0 , the linear vector field on T_*P_0/G_0 or T^*P_0/G_0 and their tangent flows is not so univocal as it has place in the finite dimensional case described by Proposition 3.4.2 of [13].

6.2 Realification and the subalgebroid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of the partial isometries

All structures from the previous sections were investigated in the framework of the category of complex (holomorphic) Banach manifolds. Passing to the underlying real Banach manifolds with underlying real structures of the Banach Lie groupoids, Banach Lie algebroids and the Banach sub Poisson manifolds one can reformulate statements of the previous sections to their real versions.

In particular case after realification one can consider $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a real Banach Lie groupoid. By $\mathcal{U}(\mathfrak{M})$ we denote the set of partial isometries of the W^* -algebra \mathfrak{M} . The inverse map $\iota : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ and the conjugation map $*$: $\mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ define the involution

$$J(x) := \iota(x)^* = \iota(x^*) \quad (6.5)$$

which is an automorphism of the real Banach Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. One easily see that $x \in \mathcal{U}(\mathfrak{M})$ iff $J(x) = x$. Since $J : \mathcal{G}(\mathfrak{M}) \rightarrow \mathcal{G}(\mathfrak{M})$ is an involutive automorphism of the real Banach Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$, see [17], we conclude that the groupoid of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a wide real Banach Lie subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

For any $p_0 \in \mathcal{L}(\mathfrak{M})$ one has the transitive subgroupoid $\mathcal{U}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ and a variant of Proposition 1.2 is valid for this case. As for $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ one has a groupoid isomorphism

$$\begin{array}{ccc} \frac{P_0^u \times P_0^u}{U_0} & \xrightarrow{\phi} & \mathcal{U}_{p_0}(\mathfrak{M}) \\ \downarrow & & \downarrow \text{t} \quad \downarrow \text{s} \\ P_0^u / U_0 & \xrightarrow{\varphi} & \mathcal{L}_{p_0}(\mathfrak{M}) \end{array}, \quad (6.6)$$

where $P_0^u := P_0 \cap \mathcal{U}(\mathfrak{M})$ and U_0 is the group of unitary elements of W^* -subalgebra $p_0 \mathfrak{M} p_0$.

In order to express J in the coordinate (1.8) we note that $J : \Omega_{p\tilde{p}} \rightarrow \Omega_{p\tilde{p}}$ and $\psi_{p\tilde{p}}(J(x)) = (y_p, J(z_{p\tilde{p}}), \tilde{y}_{\tilde{p}})$. So, $x \in \Omega_{p\tilde{p}} \cap \mathcal{U}(\mathfrak{M})$ iff $z_{p\tilde{p}} z_{p\tilde{p}}^* = p$ (or equivalently $z_{p\tilde{p}}^* z_{p\tilde{p}} = \tilde{p}$). Hence, fixing $z_{p\tilde{p}}^0 \in \mathbf{t}^{-1}(p) \cap \mathbf{s}^{-1}(\tilde{p})$ we can parametrize $z_{p\tilde{p}} = z_{p\tilde{p}}^0 g$ univocally by $g \in U(\tilde{p} \mathfrak{M} \tilde{p})$, where $U(\tilde{p} \mathfrak{M} \tilde{p})$ is the group of unitary elements of W^* -subalgebra $\tilde{p} \mathfrak{M} \tilde{p}$. A local chart on a properly choosen open subset $\Omega_{\tilde{p}} \subset U(\tilde{p} \mathfrak{M} \tilde{p})$ is given by $\log : \Omega_{\tilde{p}} \rightarrow i(\tilde{p} \mathfrak{M} \tilde{p})^h$, where $(\tilde{p} \mathfrak{M} \tilde{p})^h$ is the hermitian part of $\tilde{p} \mathfrak{M} \tilde{p}$. Summarizing the above facts we obtain the atlas of charts parametrized by $p, \tilde{p} \in \mathcal{L}(\mathfrak{M})$:

$$\psi_{p\tilde{p}}^u : \Omega_{p\tilde{p}} \cap \mathcal{U}(\mathfrak{M}) \ni x \mapsto \psi_{p\tilde{p}}^u(x) = (y_p, -i \log[(z_{p\tilde{p}}^0)^{-1} z_{p\tilde{p}}], \tilde{y}_{\tilde{p}}) \in (1-p) \mathfrak{M} p \oplus (\tilde{p} \mathfrak{M} \tilde{p})^h \oplus (1-\tilde{p}) \mathfrak{M} \tilde{p}, \quad (6.7)$$

where one consider $(1-p) \mathfrak{M} p$ and $(1-\tilde{p}) \mathfrak{M} \tilde{p}$ as a real Banach spaces. This atlas defines the structure of a real submanifold on $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$. Hence, similarly to the complex case one can apply the coordinate description to the groupoid of partial isometries.

One can investigate the Atiyah sequence, the predual Atiyah sequence and the short exact sequence of \mathcal{VB} -groupoids canonically related to the groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ but we will not investigate this subject here.

6.3 Some remarks about the case of $\mathfrak{M} = L^\infty(\mathcal{H})$

Let \mathcal{H} be a separable complex Hilbert space. By $L^\infty(\mathcal{H})$ we denote the W^* -algebra of bounded operators on \mathcal{H} . The predual Banach space $L^\infty(\mathcal{H})_*$ of $L^\infty(\mathcal{H})$ is the ideal $L^1(\mathcal{H}) \subset L^\infty(\mathcal{H})$ of the trace class operators and the pairing between $(\rho, x) \in L^1(\mathcal{H}) \times L^\infty(\mathcal{H})$ is given by

$$\langle \rho, x \rangle := \text{Tr}(\rho x). \quad (6.8)$$

The lattice $\mathcal{L}(L^\infty(\mathcal{H}))$ of orthogonal projections is canonically isomorphic with the lattice $\mathcal{L}(\mathcal{H})$ of Hilbert subspaces of \mathcal{H} .

From Banach inverse operator theorem it follows that $\mathcal{G}(L^\infty(\mathcal{H}))$ consists of operators with a closed image, e.g. $A \in \mathcal{G}(L^\infty(\mathcal{H})) \subset L^\infty(\mathcal{H})$ if and only if $ImA = \overline{ImA}$. The set of operators with the closed images let us denote by $\mathcal{G}(\mathcal{H})$. Therefore we can identify $\mathcal{G}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}(L^\infty(\mathcal{H}))$ with the groupoid $\mathcal{G}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$, where $\mathbf{s}(A) = (ker A)^\perp$ and $\mathbf{t}(A) = ImA$.

The groupoid $\mathcal{G}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$ of the partially invertible operators on \mathcal{H} splits on the groupoid $\mathcal{G}_{fin}(\mathcal{H}) \rightrightarrows \mathcal{L}_{fin}(\mathcal{H})$ of the finite-rank operators defined by

$$\mathcal{L}_{fin}(\mathcal{H}) := \bigcup_{N=1}^{\infty} \mathcal{L}_N(\mathcal{H}), \quad (6.9)$$

$$\mathcal{G}_{fin}(\mathcal{H}) := \mathbf{t}^{-1}(\mathcal{L}_{fin}(\mathcal{H})) \cap \mathbf{s}^{-1}(\mathcal{L}_{fin}(\mathcal{H})) \quad (6.10)$$

and the groupoid $\mathcal{G}_\infty(\mathcal{H}) \rightrightarrows \mathcal{L}_\infty(\mathcal{H})$ of the infinite dimensional range partially invertible operators, i.e. $A \in \mathcal{G}_\infty(\mathcal{H})$ if and only if $\dim_{\mathbb{C}} ImA = \infty$. We mention that $\mathcal{L}_N(\mathcal{H})$ consists of projections of rank N .

We define the Fredholm subgroupoid $\mathcal{G}_{Fred}(\mathcal{H}) \rightrightarrows \mathcal{L}_{Fred}(\mathcal{H})$ of the groupoid $\mathcal{G}_\infty(\mathcal{H}) \rightrightarrows \mathcal{L}_\infty(\mathcal{H})$ as follows

$$\mathcal{L}_{Fred}(\mathcal{H}) := \perp(\mathcal{L}_{fin}(\mathcal{H})) \quad (6.11)$$

$$\mathcal{G}_{Fred}(\mathcal{H}) := \mathbf{t}^{-1}(\mathcal{L}_{Fred}(\mathcal{H})) \cap \mathbf{s}^{-1}(\mathcal{L}_{Fred}(\mathcal{H})) \quad (6.12)$$

where the involution $\perp : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is defined by the orthogonal complements

$$\perp(p) = p^\perp := 1 - p \quad (6.13)$$

of $p \in \mathcal{L}(\mathcal{H})$.

Proposition 6.3. *The involution $\perp : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is an authomorphism of the complex analytic Banach manifold $\mathcal{L}(\mathcal{H})$.*

Proof. At first we show that $\perp(\Pi_p) = \Pi_{p^\perp}$. For this reason we note that the Banach splitting

$$L^\infty(\mathcal{H}) = qL^\infty(\mathcal{H}) \oplus (1 - p)L^\infty(\mathcal{H}) \quad (6.14)$$

is equivalent to the splitting

$$\mathcal{H} = q\mathcal{H} \oplus (1 - p)\mathcal{H} \quad (6.15)$$

of \mathcal{H} on the corresponding Hilbert subspaces, where $q \in \Pi_p$. Since existence of the splitting (6.15) is equivalent to the existence of the splitting

$$\mathcal{H} = (1 - q)\mathcal{H} \oplus p\mathcal{H} \quad (6.16)$$

we find that $1 - p \in \Pi_{p^\perp}$. Thus one has $\perp(\Pi_p) = \Pi_{p^\perp}$.

Now using (1.3) we obtain

$$\varphi_p^{-1}(y_p) = q = l(p + y_p) = (p + y_p)(p + y_p)^{-1}. \quad (6.17)$$

Thus and from (1.4) we find that

$$y_{p^\perp} = (\varphi_{p^\perp} \circ \perp \circ \varphi_p^{-1})(y_p) = [(1 - p)(1 - (p + y_p)(p + y_p)^{-1})]^{-1} - (1 - p) \quad (6.18)$$

for $y_p \in \varphi_p(\Pi_p)$ and $y_{p^\perp} \in \varphi_{p^\perp}(\Pi_{p^\perp})$. The above shows that \perp is a complex analytic authomorphism of $\mathcal{L}(\mathcal{H})$. \square

From the above proposition we conclude

Corrolary 6.4. *The Fredholm groupoid $\mathcal{G}_{Fred}(\mathcal{H}) \rightrightarrows \mathcal{L}_{Fred}(\mathcal{H})$ is a complex Banach Lie subgroupoid of the Banach Lie groupoid $\mathcal{G}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$.*

The all results of the previous sections concerning fibre-wise linear sub Poisson structures one can apply and investigate in the case of Banach Lie groupoids $\mathcal{G}(\mathcal{H}) \rightrightarrows \mathcal{L}(\mathcal{H})$, $\mathcal{G}_{Fred}(\mathcal{H}) \rightrightarrows \mathcal{L}_{Fred}(\mathcal{H})$ and $\mathcal{G}_{fin}(\mathcal{H}) \rightrightarrows \mathcal{L}_{fin}(\mathcal{H})$ important from geometrical as well as physical point of view. We back to this investigations in the next paper.

7 Appendix

7.1 \mathcal{VB} -groupoids

The concept of \mathcal{VB} -groupoids goes back to Pradines, [19]. It is abstracted the vector bundle structure of groupoid $TG \rightrightarrows TM$ tangent to a groupoid $G \rightrightarrows M$.

In a diagramatic presentation a \mathcal{VB} -groupoid is a structure

$$\begin{array}{ccc}
 \Omega & \begin{array}{c} \xrightarrow{\tilde{\lambda}} \\ \xleftarrow{\tilde{0}} \end{array} & \Gamma \\
 \begin{array}{c} \uparrow \parallel \\ \tilde{\mathbf{t}} \\ \downarrow \parallel \\ \mathbf{\tilde{s}} \end{array} & & \begin{array}{c} \uparrow \parallel \\ \mathbf{t} \\ \downarrow \parallel \\ \mathbf{s} \end{array} \\
 E & \begin{array}{c} \xrightarrow{\lambda} \\ \xleftarrow{0} \end{array} & M
 \end{array}, \tag{7.1}$$

in which

- (i) $\Omega \rightrightarrows E$ and $\Gamma \rightrightarrows M$ are Lie groupoids;
- (ii) $\Omega \xrightarrow{\tilde{\lambda}} \Gamma$ and $E \xrightarrow{\lambda} M$ are vector bundles;
- (iii) the above structures are subjected to the consistency conditions such that the groupoids structural maps: $\mathbf{s}, \mathbf{t}, \mathbf{1}, \tilde{\mathbf{s}}, \tilde{\mathbf{t}}, \tilde{\mathbf{1}}$ and groupoid operations, i.e. the product and the inverse map, are vector bundle morphisms;
- (iv) the vector bundle projection $\tilde{\lambda}$ and λ as well as null sections $\tilde{0}$ and 0 define groupoid morphisms;
- (v) the "double source map" $(\tilde{\lambda}, \tilde{\mathbf{s}}) : \Omega \rightarrow \Gamma \times_M E$, where $\Gamma \times_M E := \{(\gamma, e) \in \Gamma \times E : \mathbf{s}(\gamma) = \lambda(e)\}$, is surjective submersion,
- (vi) for $w_1, w_2, \nu_1, \nu_2 \in \Omega$ such that $\tilde{\mathbf{t}}(\nu_1) = \tilde{\mathbf{s}}(w_1)$, $\tilde{\mathbf{t}}(\nu_2) = \tilde{\mathbf{s}}(w_2)$, $\tilde{\lambda}(w_1) = \tilde{\lambda}(w_2)$, $\tilde{\lambda}(\nu_1) = \tilde{\lambda}(\nu_2)$ one has the condition

$$(w_1 + w_2)(\nu_1 + \nu_2) = w_1\nu_1 + w_2\nu_2$$

called the interchange law.

7.2 The dual of a \mathcal{VB} -groupoid

Here we discuss briefly the procedure of the dualization of (7.1). For use of this procedure one defines the core of a \mathcal{VB} -groupoid to be

$$K := \{\omega \in \Omega : \exists_{m \in M} \tilde{s}(\omega) = 0_m \text{ and } \tilde{\lambda}(\omega) = \mathbf{1}_m\} \quad (7.2)$$

where $0 : M \rightarrow E$ is the zero section of $E \xrightarrow{\lambda} M$ and $\mathbf{1} : M \rightarrow \Gamma$ the identity map of $\Gamma \rightrightarrows M$. Defining $\lambda_K : K \rightarrow M$ by $\lambda_K := s \circ \tilde{\lambda}$ one easily verifies that the core is a vector bundle over M . The \mathcal{VB} -groupoid

$$\begin{array}{ccc} \Omega^* & \begin{array}{c} \xrightarrow{\tilde{\lambda}_*} \\ \xleftarrow{\tilde{0}_*} \end{array} & \Gamma \\ \begin{array}{c} \uparrow \tilde{\mathbf{1}}_* \\ \parallel \tilde{\mathbf{t}}_* \\ \downarrow \tilde{s}_* \end{array} & & \begin{array}{c} \parallel \mathbf{t} \\ \uparrow \mathbf{s} \\ \uparrow \mathbf{1} \end{array} \\ K^* & \begin{array}{c} \xrightarrow{\lambda_*} \\ \xleftarrow{0_*} \end{array} & M \end{array}, \quad (7.3)$$

dual to a \mathcal{VB} -groupoid (7.1) is defined as follows.

Take $\gamma \in \Gamma_m^n := s^{-1}(m) \cap t^{-1}(n)$ and $\varphi \in \Omega_\gamma^*$, where $\Omega_\gamma := \tilde{\lambda}^{-1}(\gamma)$, then

$$\langle \tilde{s}_*(\varphi), k \rangle := \langle \varphi, -\tilde{0}_\gamma k^{-1} \rangle \quad \text{for } k \in K_m \quad (7.4)$$

and

$$\langle \tilde{\mathbf{t}}_*(\varphi), k \rangle := \langle \varphi, k \tilde{0}_\gamma \rangle \quad \text{for } k \in K_n. \quad (7.5)$$

The composition $\varphi\psi \in \Omega_{\gamma\delta}^*$ of $\varphi \in \Omega_\gamma^*$ and $\psi \in \Omega_\delta^*$ with $\tilde{s}_*(\varphi) = \tilde{\mathbf{t}}_*(\psi)$ one defines by

$$\langle \varphi\psi, wv \rangle := \langle \varphi, w \rangle + \langle \psi, v \rangle, \quad (7.6)$$

where $w \in \Omega_\gamma$ and $v \in \Omega_\delta$ satisfy $\tilde{s}(w) = \tilde{\mathbf{t}}(v)$. In order to define the identity map $\tilde{\mathbf{1}}_* : K^* \rightarrow \Omega^*$ at $\lambda \in K_m^*$ we note that $k = \omega - \tilde{\mathbf{1}}_{\tilde{\lambda}(\omega)} \in K_m$ where $m = \lambda(\tilde{s}(\omega)) = s(\tilde{\lambda}(\omega))$. Now one defines

$$\langle \tilde{\mathbf{1}}_{*\lambda}, w \rangle = \langle \tilde{\mathbf{1}}_{*\lambda}, k + \tilde{\mathbf{1}}_{\tilde{\lambda}(\omega)} \rangle := \langle \lambda, k \rangle. \quad (7.7)$$

Since the inverse map $\iota : \Omega \rightarrow \Omega$ is an automorphism of the vector bundle $\tilde{\lambda} : \Omega \rightarrow \Gamma$ one defines the inverse map $\iota_* : \Omega^* \rightarrow \Omega^*$ as

$$\langle \iota_*\varphi, w \rangle := -\langle \varphi, \iota(w) \rangle. \quad (7.8)$$

The straightforward verification shows the correctness of the above definitions.

If the \mathcal{VB} -groupoid structure of (7.1) is modelled on the reflexive Banach spaces (what has a place in the finite dimensional case) then dualizing (7.3) we obtain back the initial \mathcal{VB} -groupoid.

7.3 Short exact sequence of \mathcal{VB} -groupoids

A short exact sequence of \mathcal{VB} -groupoids

$$\begin{array}{ccccccc} \Omega_1 & \xrightarrow{\quad} & \Gamma & & \Gamma & & \Gamma \\ \parallel & \searrow F & \parallel & & \parallel & \searrow H & \parallel \\ E_1 & \xrightarrow{\quad} & M & \xrightarrow{\quad} & \Omega_2 & \xrightarrow{\quad} & \Omega_3 \\ \parallel & \searrow f & \parallel & & \parallel & \searrow h & \parallel \\ E_2 & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M \\ \parallel & & \parallel & & \parallel & & \parallel \\ E_3 & & & & & & M \end{array} \quad (7.9)$$

consists of the groupoids $\Omega_k \rightrightarrows E_k$, $k = 1, 2, 3$, having $\Gamma \rightrightarrows M$ as a side groupoid and the groupoids morphisms (F, f) and (H, h) are such that

$$\Omega_1 \xrightarrow{F} \Omega_2 \xrightarrow{H} \Omega_3 \quad (7.10)$$

is a short exact sequence of vector bundles over Γ .

The following statements are valid:

- (i) $E_1 \xrightarrow{f} E_2 \xrightarrow{h} E_3$ is a short exact sequence of vector bundles over M
- (ii) $K_1 \xrightarrow{F_K} K_2 \xrightarrow{H_K} K_3$ is a short exact sequence of vector bundles over M , where the morphisms F_K and H_K are induced by F and H respectively.
- (iii) The dualization of (7.9) results in

$$\begin{array}{ccccccc}
 \Omega_3^* & \xrightarrow{\quad} & \Gamma & \xrightarrow{\quad} & \Gamma & \xrightarrow{\quad} & \Gamma \\
 \parallel & \searrow H^* & \parallel & \searrow F^* & \parallel & \searrow & \parallel \\
 K_3^* & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M & \xrightarrow{\quad} & M \\
 & \searrow h^* & & \searrow f^* & & & \\
 & & K_2^* & \xrightarrow{\quad} & K_1^* & \xrightarrow{\quad} & M
 \end{array} \quad (7.11)$$

which is also a short exact sequence of \mathcal{VB} -groupoids.

7.4 Poisson groupoid

Let $\Gamma \rightrightarrows M$ be a Lie groupoid with a Poisson structure π on the manifold Γ . Then (Γ, π) is called a Poisson groupoid if the Poisson anchor $\pi^\# : T^*\Gamma \rightarrow T\Gamma$ is a morphism of groupoids over some map $A^*\Gamma \rightarrow TM$.

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